

THEORY OF COMPTON SCATTERING BY ANISOTROPIC ELECTRONS

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ABSTRACT

Compton scattering plays an important role in various astrophysical objects such as accreting black holes and neutron stars, pulsars, and relativistic jets, clusters of galaxies as well as the early Universe. In most of the calculations it is assumed that the electrons have isotropic angular distribution in some frame. However, there are situations where the anisotropy may be significant due to the bulk motions, or anisotropic cooling by synchrotron radiation, or anisotropic source of seed soft photons. We develop here an analytical theory of Compton scattering by anisotropic distribution of electrons that can simplify significantly the calculations. Assuming that the electron angular distribution can be represented by a second order polynomial over cosine of some angle (dipole and quadrupole anisotropy), we integrate the exact Klein-Nishina cross-section over the angles. Exact analytical and approximate formulae valid for any photon and electron energies are derived for the redistribution functions describing Compton scattering of photons with arbitrary angular distribution by anisotropic electrons. The analytical expressions for the corresponding photon scattering cross-section on such electrons as well as the mean energy of scattered photons, its dispersion and radiation pressure force are also derived. We applied the developed formalism to the accurate calculations of the thermal and kinematic Sunyaev-Zeldovich effects for arbitrary electron distributions.

Subject headings: accretion, accretion disks – cosmic background radiation – galaxies: jets – methods: analytical – radiation mechanisms: nonthermal – scattering

1. INTRODUCTION

Compton scattering is one of the most important radiative process that shapes the spectra of various sources: black holes and neutron stars in X-ray binaries, pulsars and pulsar wind nebulae, jets from active galactic nuclei, and the early Universe. Compton scattering kernel takes a simple form if electrons are ultra-relativistic with the Lorentz factor $\gamma \gg 1$ (Blumenthal & Gould 1970). In a general case, when no restrictions are made on the energies of photon and electrons, Jones (1968) derived the kernel for isotropic electrons and photons. The formulae there contain a few misprints, but even by correcting those (see e.g. Pe'er & Waxman 2005) they cannot be used for calculations because of a number of numerical cancellations (see e.g. Belmont 2009). An alternative derivation to that kernel was given by Brinkmann (1984) and Nagirner & Poutanen (1994, NP94 hereafter), who showed how to extend numerical scheme to cover all photon and electron energies of interest in astrophysical sources.

In real astrophysical environments, the radiation field does not need to be isotropic and a more general redistribution function is required to describe angle-dependent Compton scattering. NP94 extended previous results to the situation when the photon distribution can be represented as a linear function of some polar angle cosine (Eddington approximation), deriving an analytical formula for the first moment of the kernel. Aharonian & Atoyan (1981) were first to derive a redistribution function for arbitrary photon angular dependence (see also Prasad et al. 1986). Kershaw et al. (1986) and Kershaw (1987) have developed numerical methods to compute the kernel efficiently and with a high accuracy. All these works neglect the effect of photon polarization.

Nagirner & Poutanen (1993) have derived a general Compton scattering redistribution matrix for Stokes parameters as-

suming an isotropic electron distribution. A general relativistic kinetic equation incorporating the effects of induced scattering and polarization of photons as well as electron polarization and degeneracy has been derived by Nagirner & Poutanen (2001).

In this paper, we propose a method to extend previous results to the case where the electron distribution is no longer isotropic, but can have weak anisotropies which can be represented as a second order polynomial of the cosine of some polar angle. The proposed formalism can find its application in a number of astrophysical problems. The distortion of the cosmic microwave background (CMB) caused by the hot electron gas in clusters of galaxies (i.e. kinematic and thermal Sunyaev-Zeldovich effect) is an obvious application. The electron distribution, isotropic in the cluster frame, can be Lorentz transformed to the CMB frame, resulting in a dipole term linear in cluster velocity and a small quadrupole correction. Compton scattering then can be directly computed in the CMB frame. Another possible application concerns the models of outflowing accretion disk-coronae or jets (Beloborodov 1999; Malzac et al. 2001). If the outflow velocities are non-relativistic, the radiation transport can be considered directly in the accretion disk frame following recipe by Poutanen & Svensson (1996), with the Lorentz transformed electron distribution.

Weak anisotropies in the electron distribution can arise in high-energy sources with ordered magnetic field, because of the pitch angle-dependence of the synchrotron cooling rate and/or anisotropic injection of high-energy electrons (e.g. Björnsson 1985; Roland et al. 1985; Crusius-Waetzel & Lesch 1998; Schopper et al. 1998). An anisotropic electron distribution is also a very natural outcome of the photon breeding operation in relativistic jets with the toroidal magnetic field (Stern & Poutanen 2006, 2008), because the electron-positron pairs born inside the jet by the external high-energy photons move perpendicularly to the field.

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Our method is also extendable to the polarized radiation using the techniques developed in Nagirner & Poutanen (1993). It is also in principle possible to calculate the scattering redistribution function in the case when the electron distribution is expressible as an arbitrary order expansion over the polar angle cosine. Unfortunately, in the latter case the analytical expressions become extremely cumbersome and the advantage over direct numerical integration becomes small.

Although here we consider only photon scattering it is also possible to apply the same method for electrons interacting with the photons in the case when photon angular distribution is expressible as an expansion of powers of the polar angle cosine. This can be of interest only in the deep Klein-Nishina regime where the electron can lose or gain a large fraction of its initial energy in one scattering and continuous energy loss approach is not applicable.

Our paper is organized as follows. In Section 2 we introduce the relativistic kinetic equation for Compton scattering and define the redistribution function and total cross-section. The expressions for the total cross-section, the mean energy and dispersion of scattered photons, and the radiation pressure force are given in Section 3. The exact analytical formulae for the redistribution function for mono-energetic anisotropic electrons as well as approximate formulae valid in Thomson regime are derived in Section 4. We present the numerical examples of redistribution functions in Section 5, where we also develop the relativistic theory of Sunyaev-Zeldovich effect. We summarize our findings in Section 6.

2. RELATIVISTIC KINETIC EQUATION

Let us define the dimensionless photon four-momentum as $\underline{x} = \{x, \mathbf{x}\} = x\{1, \boldsymbol{\omega}\}$, where $\boldsymbol{\omega}$ is the unit vector in the photon propagation direction and $x \equiv hv/m_e c^2$. The photon distribution will be described by the occupation number n . The dimensionless electron four-momentum is $\underline{p} = \{\gamma, \mathbf{p}\} = \{\gamma, p\boldsymbol{\Omega}\} = \gamma\{1, \beta\boldsymbol{\Omega}\}$, where $\boldsymbol{\Omega}$ is the unit vector along the electron momentum, γ and $p = \sqrt{\gamma^2 - 1}$ are the electron Lorentz factor and its momentum in units of $m_e c$, and $\beta = p/\gamma$ is the electron velocity in units of speed of light. The momentum distribution of electrons is described by the relativistically invariant distribution function $f_e(\mathbf{p})$ (see Belyaev & Budker 1956; NP94).

The interaction between photons and electrons via Compton scattering (in linear approximation, i.e. ignoring induced scattering and electron degeneracy) can be described by the explicitly covariant relativistic kinetic equation for photons (Pomraning 1973; Nagirner & Poutanen 1993; NP94):

$$\underline{x} \cdot \nabla n(\mathbf{x}) = \frac{r_e^2}{2} \int \frac{d^3 p}{\gamma} \frac{d^3 p_1}{\gamma_1} \frac{d^3 x_1}{x_1} \delta^4(\underline{p}_1 + \underline{x}_1 - \underline{p} - \underline{x}) F \times [n(\mathbf{x}_1) f_e(\mathbf{p}_1) - n(\mathbf{x}) f_e(\mathbf{p})], \quad (1)$$

where $\nabla = \{\partial/\partial t, \nabla\}$ is the four-gradient, r_e is the classical electron radius, F is the Klein-Nishina reaction rate (Berestetskii et al. 1982)

$$F = \left(\frac{1}{\xi} - \frac{1}{\xi_1}\right)^2 + 2 \left(\frac{1}{\xi} - \frac{1}{\xi_1}\right) + \frac{\xi}{\xi_1} + \frac{\xi_1}{\xi}, \quad (2)$$

and

$$\xi = \underline{p}_1 \cdot \underline{x}_1 = \underline{p} \cdot \underline{x}, \quad \xi_1 = \underline{p}_1 \cdot \underline{x} = \underline{p} \cdot \underline{x}_1 \quad (3)$$

are the four-products of corresponding momenta. The second equalities in both equations (3) arise from the four-momentum

conservation law. The invariant scalar product of the photon four-momenta can be written in the laboratory frame as well as in the frame related to a specific electron

$$q \equiv \underline{x} \cdot \underline{x}_1 = x x_1 (1 - \mu) = \xi \xi_1 (1 - \mu_0) = \xi - \xi_1, \quad (4)$$

where $\mu = \boldsymbol{\omega} \cdot \boldsymbol{\omega}_1$ is the cosine of the photon scattering angle in some frame and μ_0 is the corresponding cosine in the electron rest frame.

In any frame, the kinetic equation can be also put in the usual form of the radiative transfer equation (Nagirner & Poutanen 1993):

$$\left(\frac{1}{c} \frac{\partial}{\partial t} + \boldsymbol{\omega} \cdot \nabla\right) n(\mathbf{x}) = -\sigma_T N_e \bar{s}_0(\mathbf{x}) n(\mathbf{x}) + \sigma_T N_e \frac{1}{x} \int_0^\infty x_1 dx_1 \int d^2 \omega_1 R(\mathbf{x}_1 \rightarrow \mathbf{x}) n(\mathbf{x}_1), \quad (5)$$

where N_e is the electron density in that frame. Here we have defined the photon redistribution function

$$R(\mathbf{x}_1 \rightarrow \mathbf{x}) = \frac{3}{16\pi} \frac{1}{N_e} \int \frac{d^3 p}{\gamma} \frac{d^3 p_1}{\gamma_1} f_e(\mathbf{p}_1) F \delta^4(\underline{p}_1 + \underline{x}_1 - \underline{p} - \underline{x}) \quad (6)$$

and the total scattering cross-section (in units of Thomson cross-section σ_T)

$$\bar{s}_0(\mathbf{x}) = \frac{3}{16\pi} \frac{1}{N_e} \frac{1}{x} \int \frac{d^3 p}{\gamma} \frac{d^3 p_1}{\gamma_1} \frac{d^3 x_1}{x_1} f_e(\mathbf{p}) F \delta^4(\underline{p}_1 + \underline{x}_1 - \underline{p} - \underline{x}). \quad (7)$$

2.1. Electron distribution and scattering geometry

Let us now consider a specific frame E . Our basic assumption is that the anisotropy of the electron distribution in this frame can be expressed as a second order polynomial expansion in the cosine of the polar angle in some coordinate system $(\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3)$:

$$\frac{1}{N_e} f_e(\mathbf{p}) = f_e(\gamma, \eta_e) = \sum_{k=0}^2 f_k(\gamma) P_k(\eta_e), \quad (8)$$

where N_e is the electron density in frame E , $\eta_e = \boldsymbol{\Omega} \cdot \mathbf{l}_3$ is the cosine of the polar angle of the electron momentum, $P_k(\eta_e)$ are the Legendre polynomials, and we now switched to the dimensionless distribution function $f_e(\gamma, \eta_e)$ normalized to unity:

$$\int d^2 \Omega \int f_e(\gamma, \eta_e) p^2 dp = 1. \quad (9)$$

The moments f_0 , f_1 and f_2 determine the energy spectrum of electrons and the relative magnitudes of the isotropic and anisotropic components. The distribution function for mono-energetic electrons of energy γ_0 can be described by

$$f_e(\gamma, \eta_e) = \frac{1}{4\pi p \gamma} \delta(\gamma - \gamma_0) \left[1 + \frac{f_1}{f_0} \eta_e + \frac{f_2}{f_0} P_2(\eta_e) \right], \quad (10)$$

where the ratios f_1/f_0 and f_2/f_0 are constants.

The directions of photons in this coordinate system (see Fig. 1) are given by

$$\boldsymbol{\omega} = \sqrt{1 - \eta^2} \cos \phi \mathbf{l}_1 + \sqrt{1 - \eta^2} \sin \phi \mathbf{l}_2 + \eta \mathbf{l}_3, \quad (11)$$

$$\boldsymbol{\omega}_1 = \sqrt{1 - \eta_1^2} \cos \phi_1 \mathbf{l}_1 + \sqrt{1 - \eta_1^2} \sin \phi_1 \mathbf{l}_2 + \eta_1 \mathbf{l}_3, \quad (12)$$

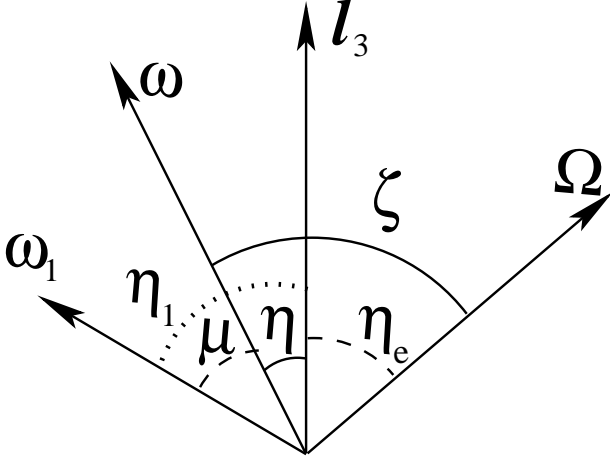


FIG. 1.— The scattering particles' momentum vectors and the vector l_3 . Note that the shown angular variables are the cosines of the respective angles.

so that the cosine of the scattering angle is

$$\mu = \omega \cdot \omega_1 = \eta\eta_1 + \sqrt{1-\eta^2}\sqrt{1-\eta_1^2}\cos(\phi-\phi_1). \quad (13)$$

3. TOTAL CROSS SECTION AND MEAN POWERS OF ENERGIES

3.1. Total cross-section

Let us simplify the expression for the total cross-section. We follow here the approach described in NP94. We rewrite the cross-section as

$$\overline{s_0}(x) = \frac{1}{x} \int s_0(\xi) \xi f_e(\gamma, \eta_e) \frac{d^3 p}{\gamma}, \quad (14)$$

where

$$s_0(\xi) = \frac{3}{16\pi} \frac{1}{\xi} \int \frac{d^3 p_1}{\gamma_1} \frac{d^3 x_1}{x_1} F \delta^4(\underline{p}_1 + \underline{x}_1 - \underline{p} - \underline{x}). \quad (15)$$

Using the identity

$$\delta(\gamma_1 + x_1 - \gamma - x) = \gamma_1 \delta(\underline{x}_1 \cdot (\underline{p} + \underline{x}) - \underline{x} \cdot \underline{p}) \quad (16)$$

and taking the integral over \underline{p}_1 in equation (15), we get

$$\begin{aligned} s_0(\xi) &= \frac{3}{16\pi} \frac{1}{\xi} \int \frac{d^3 x_1}{x_1} F \delta(\underline{x}_1 \cdot (\underline{p} + \underline{x}) - \underline{x} \cdot \underline{p}) \\ &= \frac{3}{16\pi} \frac{1}{\xi} \int \xi_1 d\xi_1 d\mu_0 d\phi_0 F \delta[\xi_1 + \xi\xi_1(1-\mu_0) - \xi] \\ &= \frac{3}{8\xi^2} \int_{\xi/(1+2\xi)}^{\xi} F d\xi_1, \end{aligned} \quad (17)$$

where we used invariant $x_1 dx_1 d^2 \omega_1 = \xi_1 d\xi_1 d\mu_0 d\phi_0$ and the fact that F does not depend on azimuthal angle ϕ_0 . Substituting F from Equation (2) we get (Berestetskii, Lifshitz, & Pitaevskii 1982; NP94)

$$s_0(\xi) = \frac{3}{8\xi^2} \left[4 + \left(\xi - 2 - \frac{2}{\xi} \right) \ln(1+2\xi) + 2\xi^2 \frac{1+\xi}{(1+2\xi)^2} \right]. \quad (18)$$

Putting $\xi \rightarrow x$, we, of course, get the total Klein-Nishina cross-section for a photon of energy x on electrons at rest.

To obtain the total scattering cross-section on an anisotropic electron distribution, we have to calculate the angular integrals over incoming electron directions in Equation (14). We

introduce the cosines between electron momentum and photons:

$$\zeta = \Omega \cdot \omega, \quad \zeta_1 = \Omega \cdot \omega_1 \quad (19)$$

so that

$$\xi = x(\gamma - p\zeta), \quad \xi_1 = x_1(\gamma - p\zeta_1). \quad (20)$$

We choose the spherical coordinate system and measure the polar angle from the direction of the *initial* photon ω , we get

$$\overline{s_0}(x) = \overline{s_0}(x, \eta) = \frac{1}{x} \int_0^\infty \frac{p^2 dp}{\gamma} \int_{-1}^1 d\zeta s_0(\xi) \xi \int_0^{2\pi} d\Phi f_e(\gamma, \eta_e). \quad (21)$$

Azimuth Φ is now defined as the difference between the azimuths of the electron momentum direction and the vector l_3 in a frame with z -axis along ω . The angular variable η_e in the expansion (8) is expressed in this frame as

$$\eta_e = \eta\zeta + \sqrt{1-\eta^2}\sqrt{1-\zeta^2}\cos\Phi. \quad (22)$$

The physical meaning of Equation (21) can be also understood if we consider a monoenergetic beam of electrons along l_3 axis: $f_e(\gamma, \eta_e) = \delta(\eta_e - 1)\delta(\gamma - \gamma_0)/(2\pi p\gamma)$. Then

$$\overline{s_0}(x, \eta) = (1 - \beta\eta) s_0(x'), \quad (23)$$

where $x' = x\gamma(1 - \beta\eta)$ is the photon energy in the electron rest frame (we omitted subscript 0 in γ and β). The factor $1 - \beta\eta$ in Equation (23) accounts for the reduced number of collision per unit length.

When calculating the azimuthal integral in equation (21) we just have to integrate $P_k(\eta_e)$ with η_e given by Equation (22). The properties of the Legendre polynomials give us the average

$$\overline{P_k(\eta_e)} = P_k(\eta) P_k(\zeta), \quad (24)$$

so that the averaged distribution function becomes

$$\overline{f_e}(\gamma, \eta, \zeta) = \frac{1}{2\pi} \int_0^{2\pi} d\Phi f_e(\gamma, \eta_e) = \sum_{k=0}^2 f_k(\gamma) P_k(\eta) P_k(\zeta). \quad (25)$$

The cross-section now can be written as

$$\overline{s_0}(x, \eta) = 4\pi \sum_{k=0}^2 P_k(\eta) \int_1^\infty p\gamma d\gamma f_k \Delta_{0k}(x, \gamma), \quad (26)$$

where

$$\Delta_{0k}(x, \gamma) = \frac{1}{2\gamma x} \int_{-1}^1 P_k(\zeta) \xi s_0(\xi) d\zeta. \quad (27)$$

Changing the integration variable and expressing P_k through ξ using Equation (20), we get

$$\Delta_{0k}(x, \gamma) = \sum_{n=0}^k b_{nk} \chi_{0n}, \quad (28)$$

where

$$\chi_{0n}(x, \gamma) = \frac{1}{2\gamma p x^{2+n}} \int_{x(\gamma-p)}^{x(\gamma+p)} \xi^{n+1} s_0(\xi) d\xi. \quad (29)$$

and

$$b_{00} = 1, \quad b_{01} = \frac{\gamma}{p}, \quad b_{11} = -\frac{1}{p}, \quad (30)$$

$$b_{02} = \frac{1}{2p^2} (2\gamma^2 + 1), \quad b_{12} = -\frac{3\gamma}{p^2}, \quad b_{22} = \frac{3}{2p^2}.$$

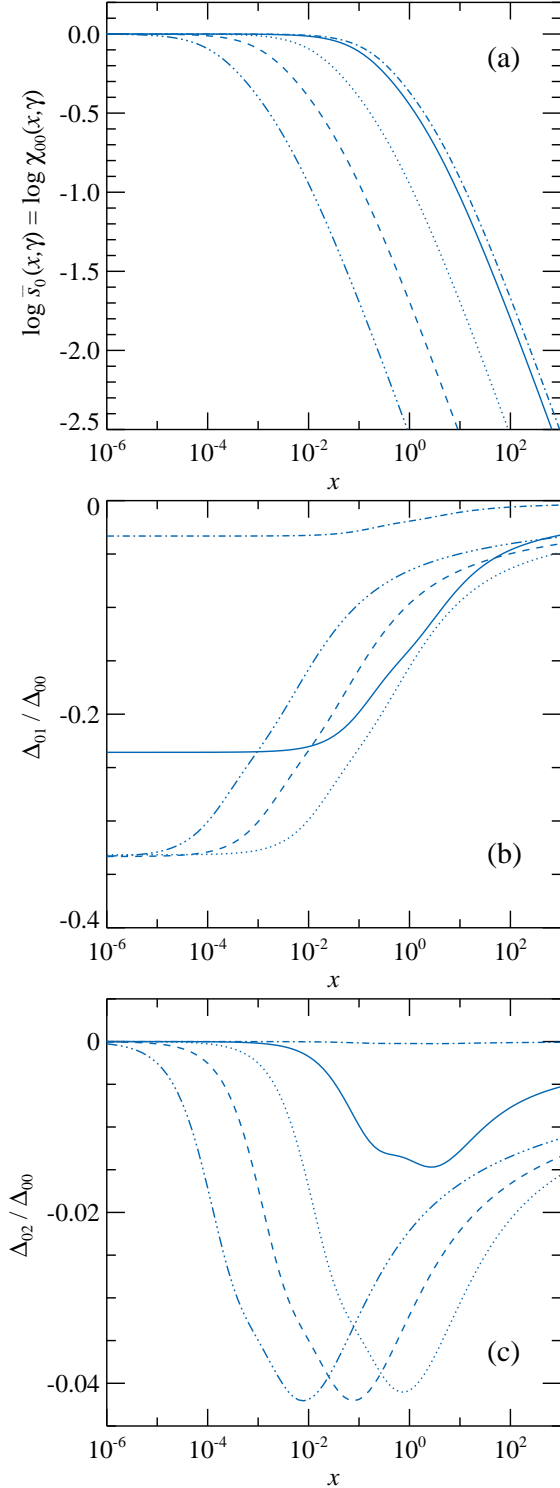


FIG. 2.— (a) Total cross-section $\chi_{00} = \Delta_{00}$ for isotropic mono-energetic electrons of various momenta $p = 0.1, 1, 10, 10^2, 10^3$ (from top to bottom, dot-dashed, solid, dotted, dashed, and triple-dot-dashed curves) as a function of photon energy x . (b) Relative correction to the cross-section arising from the dipole term in the electron distribution (8) for the same p as in panel (a). At small x the curves approach the asymptotic value in Thomson limit $-\beta/3$. (c) Relative correction to the cross-section arising from the quadrupole term Δ_{02}/Δ_{00} .

The zeroth function $\Delta_{00} = \chi_{00}$ coincides with the function $\Psi_0(x, \gamma)$ from NP94. When electron distribution is isotropic (i.e. $f_1 = f_2 = 0$), expression (26) for the total cross-section is reduced to equation (3.4.1) from NP94 and the dependence on η obviously disappears. Functions χ_{0n} of two arguments can be presented through the functions of one argument:

$$\chi_{0n}(x, \gamma) = \frac{1}{2\gamma p} \frac{u^{2+n}}{2+n} \psi_{n+1,0}(xu) \Big|_{u=\gamma-p}^{u=\gamma+p}, \quad (31)$$

where

$$\psi_{i0}(\xi) = \frac{i+1}{\xi^{i+1}} \int_0^\xi t^i s_0(t) dt. \quad (32)$$

The explicit expressions for the functions $\psi_{i0}(\xi)$ ($i = 1, 2, 3$) can be found in Appendix B (see also NP94). Thus the total cross-section is given by a single integral over the electron energy (26) with all functions under the integral given by analytical expressions. Numerical calculations of functions Δ_{0k} can be separated into three regimes: (1) in Thomson regime, $x\gamma \ll 1$, the series expansion (see Appendix D) can be used; (2) for $p \ll 1$, but x not sufficiently small for regime (1), we numerically take the integral in Equation (27) using 5-point Gaussian quadrature to reach accuracy better than 1%; (3) in other cases, we use the sum in Equation (28) and analytical expressions (31) for χ_{0n} .

For mono-energetic electron distribution of Lorentz factor γ given by Equation (10), we can introduce the cross-section analogously to Equation (26):

$$\overline{s}_0(x, \gamma, \eta) = \sum_{k=0}^2 \frac{f_k}{f_0} P_k(\eta) \Delta_{0k}(x, \gamma). \quad (33)$$

For isotropic mono-energetic electrons, the total cross-section is shown in Figure 2a. The relative corrections arising due to the dipole and quadrupole term in the electron distribution are shown in Figures 2b and 2c, respectively. These have to be multiplied by the angle- and, possibly, the energy-dependent factor $P_k(\eta)f_k/f_0$ to obtain the final correction. In the Thomson limit, at small $x\gamma \ll 1$, the cross-section takes the form (see Appendix D)

$$\overline{s}_0(x, \gamma, \eta) \approx 1 - \frac{1}{3} \frac{f_1}{f_0} \eta \beta, \quad (34)$$

where the correction to unity term can be easily obtained by averaging the transport cross-section over electron directions (i.e. integrating $(1 - \beta \zeta) \eta_e$ over the angles). This corresponds to the flattening in Figure 2b at $\Delta_{01}/\Delta_{00} = -\beta/3$. The correction from the quadrupole term in this regime as well as for non-relativistic electrons becomes negligible:

$$\Delta_{02}/\Delta_{00} \approx -\frac{4}{15} \beta^2 (x\gamma). \quad (35)$$

3.2. Mean powers of scattered photon energy

In some situations, the full relativistic kinetic equations can be substituted by the approximate one obtained in Fokker-Planck approximation. This requires knowledge of various moments of the redistribution function, such as total cross-section, the mean energy and dispersion of the scattered photons (see NP94, Vurm & Poutanen 2009). It is often time-consuming to compute numerically the integrals of the redistribution function and instead direct calculations of the moments are preferable. Below we obtain analytical expressions

for the mean energy and dispersion of the energy of scattered photons in frame E as a function of the initial photon energy x and the direction of its propagation relative to a symmetry axis of the electron distribution l_3 .

Following NP94, we define the mean of powers of energy of scattered photons:

$$\overline{x_1^j s_0(x)} = \frac{1}{x} \int \langle x_1^j \rangle s_0(\xi) \xi f_e(\gamma, \eta_e) \frac{d^3 p}{\gamma}, \quad (36)$$

where now

$$\begin{aligned} \langle x_1^j \rangle s_0(\xi) &= \frac{3}{16\pi} \frac{1}{\xi} \int \frac{d^3 p_1}{\gamma_1} \frac{d^3 x_1}{x_1} F x_1^j \delta^4(\underline{p}_1 + \underline{x}_1 - \underline{p} - \underline{x}) \\ &= \frac{3}{16\pi} \frac{1}{\xi} \int x_1^{j+1} dx_1 d\omega_1 F \delta\{x_1[\gamma + x - \omega_1 \cdot (p\Omega + x\omega)] - \xi\}. \end{aligned} \quad (37)$$

3.2.1. Averaging over photon directions

Quantities (37) are not invariants (except for $j = 0$), and we have to compute the scattered photon energy in a certain frame, which we choose to be frame E . Because of the additional term x_1^j under the integral, a simple change of variables to the electron rest frame as in Equation (17) is not possible. Instead, we use the δ -function to take the integral over x_1 :

$$\langle x_1^j \rangle s_0(\xi) = \frac{3}{16\pi} \frac{1}{\xi^2} \int x_1^{j+2} d^2 \omega_1 F. \quad (38)$$

Now we change the variables to those in the electron rest frame (with subscript 0). We choose the coordinate system with the polar axis along the direction of the incoming photon ω_0 . In this frame, the cosine of the angle between the electron momentum and the incoming photon is ζ_0 . The cosine of the angle between the outgoing photon momentum and the electron is then $\zeta_{10} = \zeta_0 \mu_0 + \sqrt{1 - \zeta_0^2} \sqrt{1 - \mu_0^2} \cos \phi_0$.

We use invariants $x_1^2 d^2 \omega_1 = \xi_1^2 d\mu_0 d\phi_0$ and the energy conservation law in the electron rest frame $\xi_1 = \xi/(1 + \xi[1 - \mu_0])$, to get (see NP94)

$$x_1^2 d^2 \omega_1 = d\xi_1 d\phi_0. \quad (39)$$

Finally, we have

$$\langle x_1^j \rangle s_0(\xi) = \frac{3}{16\pi} \frac{1}{\xi^2} \int_{\xi/(1+2\xi)}^{\xi} F d\xi_1 \int_0^{2\pi} x_1^j d\phi_0. \quad (40)$$

Because ξ_1 is the energy of scattered photon in the electron rest frame, the Doppler effect gives us

$$x_1 = \xi_1(\gamma + p\zeta_{10}) = \xi_1 \left(\gamma + p\zeta_0 \mu_0 + p\sqrt{1 - \zeta_0^2} \sqrt{1 - \mu_0^2} \cos \phi_0 \right), \quad (41)$$

where we now can substitute

$$\mu_0 = 1 + \frac{1}{\xi} - \frac{1}{\xi_1}, \quad p\zeta_0 = \frac{x}{\xi} - \gamma, \quad (42)$$

which are consequences of the conservation law and of the Lorentz transformation $x = \xi(\gamma + p\zeta_0)$, respectively. The terms containing a linear combination of square roots and $\cos \phi_0$ will disappear after averaging over ϕ_0 .

We now introduce moments of the invariant cross-section

$$s_j(\xi) = \frac{3}{8\xi^{j+2}} \int_{\xi/(1+2\xi)}^{\xi} \xi_1^j F d\xi_1. \quad (43)$$

For $j = 0$, we get of course the total cross-section s_0 given by Equation (18). NP94 derived the corresponding expressions for $j = 1, 2$:

$$s_1(\xi) = \frac{3}{8\xi^3} \left(l_\xi + \frac{4}{3} \xi^2 - \frac{3}{2} \xi - \frac{\xi}{2} R_\xi - \frac{\xi^2}{3} R_\xi^3 \right), \quad (44)$$

$$s_2(\xi) = \frac{R_\xi}{16} (9 + R_\xi + 3 R_\xi^2 + 3 R_\xi^3), \quad (45)$$

where $l_\xi = \ln(1 + 2\xi)$, and $R_\xi = 1/(1 + 2\xi)$.

For the mean energy of the scattered photon we have then

$$\langle x_1 \rangle s_0(\xi) = \xi [\gamma S_1(\xi) + x S_2(\xi)], \quad (46)$$

and for the mean square of energy

$$\langle x_1^2 \rangle s_0(\xi) = \gamma^2 \xi^2 S_4(\xi) - \gamma x \xi S_5(\xi) + x^2 S_6(\xi) - \xi^2 S_7(\xi), \quad (47)$$

where

$$\begin{aligned} S_1(\xi) &= [s_0(\xi) - s_1(\xi)]/\xi, & S_2(\xi) &= [s_1(\xi) - S_1(\xi)]/\xi, \\ S_3(\xi) &= [s_1(\xi) - s_2(\xi)]/\xi, & S_4(\xi) &= [S_1(\xi) - S_3(\xi)]/\xi, \\ S_5(\xi) &= 3 S_4(\xi) - 4 S_3(\xi), & S_7(\xi) &= S_3(\xi) - S_4(\xi)/2, \\ S_6(\xi) &= s_2(\xi) - 3 S_7(\xi). \end{aligned} \quad (48)$$

All functions $S_j(\xi)$ are elementary. In addition, they are defined in such a way so that not to become zero at $\xi = 0$. The series expansion of functions s and S for small arguments are presented in Appendix A.

3.2.2. Averaging over electron directions

We need to integrate in Equation (36) over anisotropic electron distribution. We follow the derivation of the total cross-section that lead from Equation (21) to Equation (26). Representing integral over electron momentum $d^3 p = p^2 dp d\zeta d\Phi$ we get:

$$\overline{x_1^j s_0(x, \eta)} = 4\pi x^j \sum_{k=0}^2 P_k(\eta) \int_1^\infty p \gamma d\gamma f_k \Delta_{jk}(x, \gamma), \quad (49)$$

where

$$\Delta_{jk}(x, \gamma) = \frac{1}{2\gamma x^{1+j}} \int_{-1}^1 P_k(\zeta) \xi \langle x_1^j \rangle s_0(\xi) d\zeta = \sum_{n=0}^k b_{nk} \chi_{jn} \quad (50)$$

and

$$\chi_{jn}(x, \gamma) = \frac{1}{2\gamma p x^{2+j+n}} \int_{x(\gamma-p)}^{x(\gamma+p)} \langle x_1^j \rangle s_0(\xi) \xi^{n+1} d\xi. \quad (51)$$

Functions $\chi_{j0} = \Delta_{j0}$ coincide with functions $\Psi_j(x, \gamma)$ introduced by NP94, while functions χ_{0n} are given by Equation (31). The explicit expressions for the function χ_{jn} for $j = 1, 2$ (which are analogous to functions Ψ_1 and Ψ_2 from NP94) can be obtained using expression for mean powers of energies (46) or (47):

$$\chi_{1n}(x, \gamma) = \frac{1}{2\gamma p} \frac{u^{3+n}}{3+n} \left[\gamma \Psi_{2+n,1}(xu) + x \Psi_{2+n,2}(xu) \right] \Big|_{u=\gamma-p}^{u=\gamma+p} \quad (52)$$

$$\begin{aligned} \chi_{2n}(x, \gamma) &= \frac{1}{2\gamma p} \left[\gamma^2 \frac{u^{4+n}}{4+n} \Psi_{3+n,4}(xu) - \gamma \frac{u^{3+n}}{3+n} \Psi_{2+n,5}(xu) \right. \\ &\quad \left. + \frac{u^{2+n}}{2+n} \Psi_{1+n,6}(xu) - \frac{u^{4+n}}{4+n} \Psi_{3+n,7}(xu) \right] \Big|_{u=\gamma-p}^{u=\gamma+p}, \end{aligned} \quad (53)$$

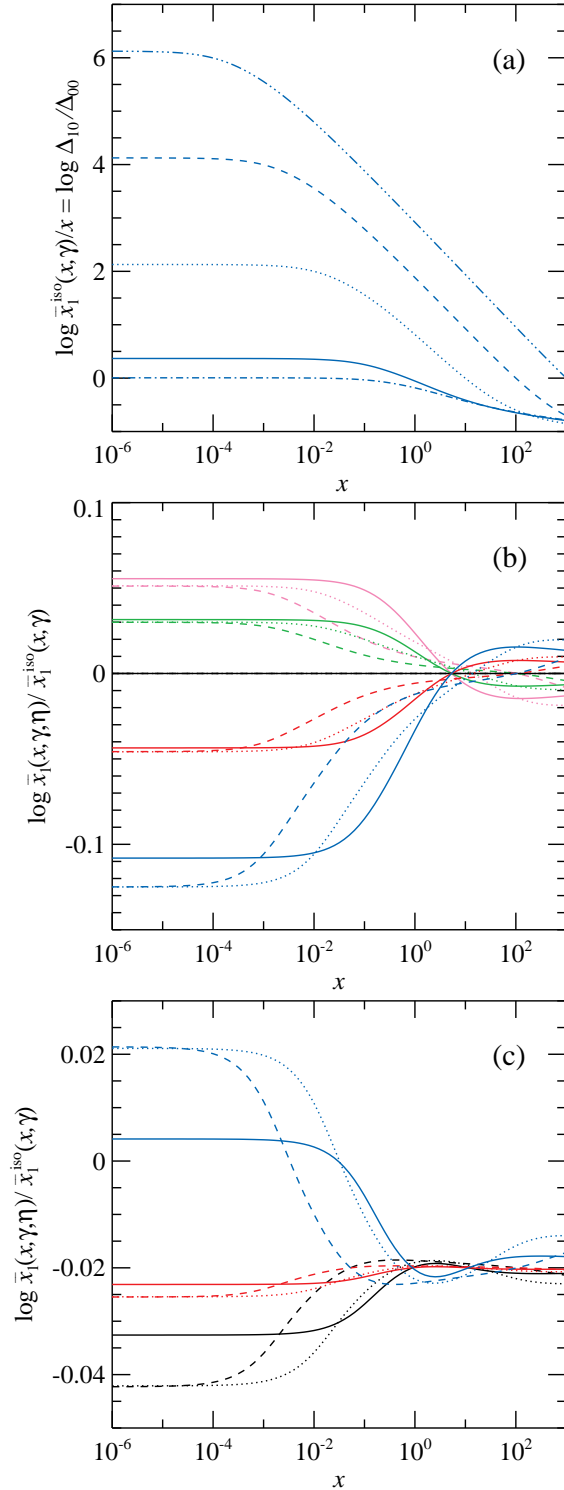


FIG. 3.— (a) Mean energy of scattered photons in units of the incident photon energy $\bar{x}_1^{\text{iso}}(x, \gamma)/x = \Delta_{10}/\Delta_{00}$ as a function of x for isotropic mono-energetic electrons of various momenta $p = 0.1, 1, 10, 10^2, 10^3$ (from bottom to top, dot-dashed, solid, dotted, dashed, and triple-dot-dashed curves). The asymptotic value at small x in Thomson approximation is $1 + \frac{4}{3}p^2$. (b) A correction to the mean energy (in units of \bar{x}_1^{iso}) arising from the linear term in the electron distribution (8) with $f_1/f_0 = 1$. Solid, dotted and dashed curves correspond for $p = 1, 10, 100$, respectively. The curves from top to bottom correspond to $\eta = -1, -0.5, 0, 0.5, 1$. At small x , the curves approach the limiting value given by Equation (58). (c) Same as (b), but for the quadrupole term in the electron distribution (8) with $f_2/f_0 = 1$. These are even functions of η , the curves from the bottom to the top correspond to $\eta = 0, 0.5, 1$. At small x , the curves approach the limiting value given by Equation (59).

where

$$\Psi_{ij}(\xi) = \frac{i+1}{\xi^{i+1}} \int_0^\xi t^i S_j(t) dt. \quad (54)$$

These are related to functions

$$\psi_{ij}(\xi) = \frac{i+1}{\xi^{i+1}} \int_0^\xi t^i s_j(t) dt, \quad (55)$$

because functions S_j are expressed through s_j . The explicit expressions for both type of these functions as well as their series expansions for small arguments are given in Appendices B.

As in the case of functions Δ_{0k} , for calculating Δ_{jk} , we consider three regimes: (1) $x\gamma \ll 1$, when we use the series expansion (see Appendix D); (2) for $p \ll 1$ we numerically take the integral in Equation (50) using Gaussian quadrature; (3) in other cases, we use the sum in Equation (50) and analytical expressions for χ_{jn} .

For mono-energetic electron distribution (10) of Lorentz factor γ , we can introduce the mean powers of photon energy analogously to Equation (49):

$$\bar{x}_1^j \bar{s}_0(x, \gamma, \eta) = x^j \sum_{k=0}^2 \frac{f_k}{f_0} P_k(\eta) \Delta_{jk}. \quad (56)$$

The mean energy of scattered photons for such electrons for a scattering act is given by the ratio of Eqs. (56) and (33). It is shown in Figure 3a. In the low-energy (Thomson) limit the energy gain factor is given by a well known expression $\bar{x}_1^{\text{iso}}/x = 1 + 4p^2/3$, which translated to $\frac{4}{3}\gamma^2$ at large γ . The relative corrections arising due to the dipole and quadrupole term in the electron distribution are shown in Figures 3b and 3c, respectively. Using asymptotic expansions of Δ_{jk} in the Thomson limit (see Appendix D), we get the asymptotic value

$$\frac{\bar{x}_1(x, \gamma, \eta)}{\bar{x}_1^{\text{iso}}(x, \gamma)} = \frac{\gamma^2 \left(1 + \frac{1}{3}\beta^2\right) - \frac{2}{3}\beta\gamma^2 \frac{f_1}{f_0} \eta + \frac{2}{15}\beta^2\gamma^2 \frac{f_2}{f_0} P_2(\eta)}{\gamma^2 \left(1 + \frac{1}{3}\beta^2\right) \left(1 - \frac{1}{3}\beta \frac{f_1}{f_0} \eta\right)}. \quad (57)$$

Thus in non-relativistic limit $\beta \ll 1$, the correction is negligible. In the relativistic limit $\gamma \gg 1$, the relative corrections arising from the two terms are

$$\frac{\bar{x}_1(x, \gamma, \eta)}{\bar{x}_1^{\text{iso}}(x, \gamma)} = \frac{1 - \frac{1}{2} \frac{f_1}{f_0} \eta}{1 - \frac{1}{3} \frac{f_1}{f_0} \eta}, \quad (58)$$

$$\frac{\bar{x}_1(x, \gamma, \eta)}{\bar{x}_1^{\text{iso}}(x, \gamma)} = 1 + \frac{1}{10} \frac{f_2}{f_0} P_2(\eta). \quad (59)$$

3.3. Energy exchange and dispersion

The difference of the photon energies before and after scattering $x - x_1$ is of course just the energy transfer to the electron gas. For the fixed angle between electrons and incident photons ζ (and fixed electron energy γ), the energy loss averaged over the directions of scattered photons is $x - \langle x_1 \rangle$. The product $(x - \langle x_1 \rangle) N_e \sigma_T s_0(\xi)$ is then the energy loss on a unit length. From Equation (46) we can easily get (see also NP94):

$$(x - \langle x_1 \rangle) s_0(\xi) = x s_0(\xi) - \gamma \xi S_1(\xi) - x \xi S_2(\xi) = (x + x\xi - \gamma\xi) S_1(\xi). \quad (60)$$

The corresponding energy loss (per unit length and in units $N_e \sigma_T$) averaged over the electron directions (and integrated

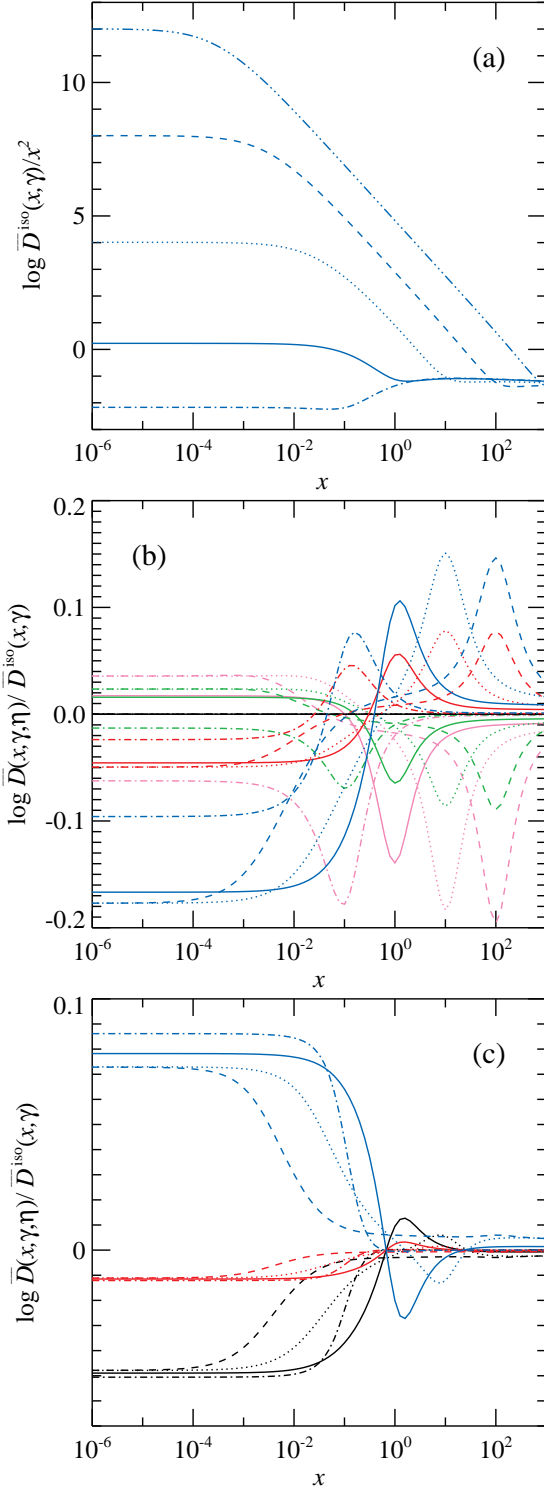


FIG. 4.— (a) Dispersion of the energy of scattered photons (in units of x^2) for isotropic mono-energetic electrons of $p = 0.1, 1, 10, 10^2, 10^3$ (from bottom to top). The asymptotic value of \bar{D}/x^2 at small x in Thomson approximation is $\frac{2}{45}(23\gamma^2 - 8)p^2$. (b) A correction to the dispersion (in terms of the isotropic quantity) arising from the linear term in the electron distribution (8) with $f_1/f_0 = 1$. Solid, dotted and dashed curves correspond for $p = 1, 10, 100$, respectively. The curves from top to bottom correspond to $\eta = -1, -0.5, 0, 0.5, 1$. (c) A correction to the dispersion arising from the quadrupole term in the electron distribution (8) with $f_2/f_0 = 1$ for the same p and η as in panel (b). These are even functions of η , the curves from bottom to top correspond to $\eta = 0, 0.5, 1$.

over electron energies) becomes [see Eqs. (26) and (49)]:

$$(x - \bar{x}_1) \bar{s}_0(x, \eta) = 4\pi x \sum_{k=0}^2 P_k(\eta) \int_1^\infty p\gamma d\gamma f_k (\Delta_{0k} - \Delta_{1k}). \quad (61)$$

The heating rate per unit volume is then

$$\dot{E} = N_e \sigma_T \int dx \int d^2\omega I(x, \omega) \left(1 - \frac{\bar{x}_1}{x}\right) \bar{s}_0(x, \eta), \quad (62)$$

where $I(x, \omega)$ is the specific intensity of radiation in a given direction. This expression can be positive (so called Compton heating) when the photons typically have larger energies than the electron gas, or negative (Compton cooling) when one considers cooling of the relativistic electron gas by soft radiation.

The dispersion of the scattered photon energy is given by the usual expression $\bar{D}(x) = \bar{x}_1^2 - \bar{x}_1^2$, which of course depends on the electron momentum distribution. For mono-energetic electrons we can define the dispersion as

$$\bar{D}(x, \gamma, \eta) = \bar{x}_1^2(x, \gamma, \eta) - \bar{x}_1^2(x, \gamma, \eta), \quad (63)$$

where $\bar{x}_1^j(x, \gamma, \eta)$ are given by Equation (56). The dispersion for isotropic electrons is shown in Figure 3a. The low-energy (Thomson) limit for $x \ll 1/\gamma$ is (see NP94)

$$\bar{D}(x, \gamma) = x^2 \frac{2}{45} (23\gamma^2 - 8) p^2. \quad (64)$$

The relative corrections arising due to the dipole and quadrupole term in the electron distribution reach about 50 per cent and are shown in Figures 4b and 4c, respectively.

3.4. Radiation force

Now we would like to derive analytic expression for the radiation force acting on the electron gas. NP94 have developed a formalism appropriate for isotropic electron distribution, when the averaged transferred momentum is along the momentum of the incoming photons, because of the symmetry. For the electron distribution described by Equation (8), the momentum is transferred in the plane containing the initial photon momentum and the symmetry axis l_3 . If the incident photons are axially symmetric around l_3 , then obviously, the total momentum transferred to the electrons has to be parallel to l_3 by symmetry. We derive here more general formulae for the total momentum transferred by a beam of photons propagating along direction ω (such as $\omega \cdot l_3 = \eta$), as well as its projections to l_3 and perpendicular direction.

Let us introduce the vector basis:

$$e_1(\omega, l_3) = \frac{l_3 - \eta \omega}{\sqrt{1 - \eta^2}}, \quad e_2(\omega, l_3) = \frac{\omega \times l_3}{\sqrt{1 - \eta^2}}, \quad e_3 = \omega. \quad (65)$$

In a single scattering act the momentum transferred is

$$\mathbf{Q} = x\omega - x_1\omega_1. \quad (66)$$

The components of the momentum transferred to the electron gas along and perpendicular to ω are:

$$Q_3 = x - x_1 \omega_1 \cdot e_3 = x - x_1 \mu, \quad (67)$$

$$Q_1 = -x_1 \omega_1 \cdot e_1 = -x_1 \frac{\eta_1 - \eta \mu}{\sqrt{1 - \eta^2}}. \quad (68)$$

Analogously to Equation (36), we define the mean transferred momentum as

$$\bar{\mathbf{Q}} \bar{s}_0(\mathbf{x}) = \frac{1}{x} \int \langle \mathbf{Q} \rangle_{s_0(\xi)} \xi f_e(\gamma, \eta_e) \frac{d^3 p}{\gamma}, \quad (69)$$

where

$$\begin{aligned} \langle \mathbf{Q} \rangle_{s_0(\xi)} &= \frac{3}{16\pi} \frac{1}{\xi} \int \frac{d^3 p_1}{\gamma_1} \frac{d^3 x_1}{x_1} F \mathbf{Q} \delta^4(\underline{p}_1 + \underline{x}_1 - \underline{p} - \underline{x}) \\ &= \frac{3}{16\pi} \frac{1}{\xi} \int x_1 dx_1 d^2 \omega_1 F \mathbf{Q} \delta\{x_1[\gamma + x - \omega_1 \cdot (\mathbf{p} \boldsymbol{\Omega} + x \boldsymbol{\omega})] - \xi\}. \end{aligned} \quad (70)$$

3.4.1. Averaging over photon directions

Let us introduce a vector basis defined by the photon and electron momenta:

$$\mathbf{e}_1(\boldsymbol{\omega}, \boldsymbol{\Omega}) = \frac{\boldsymbol{\Omega} - \zeta \boldsymbol{\omega}}{\sqrt{1 - \zeta^2}}, \quad \mathbf{e}_2(\boldsymbol{\omega}, \boldsymbol{\Omega}) = \frac{\boldsymbol{\omega} \times \boldsymbol{\Omega}}{\sqrt{1 - \zeta^2}}, \quad \mathbf{e}_3 = \boldsymbol{\omega}, \quad (71)$$

where $\zeta = \boldsymbol{\omega} \cdot \boldsymbol{\Omega}$, therefore $\boldsymbol{\Omega} = \sqrt{1 - \zeta^2} \mathbf{e}_1(\boldsymbol{\omega}, \boldsymbol{\Omega}) + \zeta \mathbf{e}_3$. Fixing the angle $\arccos \zeta$ between the electrons of momentum $\mathbf{p} \boldsymbol{\Omega}$ and the incident photon momentum and averaging over directions of scattered photons, we can get the mean momentum transmitted in the direction $\boldsymbol{\omega}$:

$$\langle Q_3 \rangle_{s_0(\xi)} = \langle x - x_1 \rangle_{s_0(\xi)} + \langle x_1 (1 - \mu) \rangle_{s_0(\xi)}. \quad (72)$$

The first term is given by Equation (60), the second term is

$$\begin{aligned} \langle x_1 (1 - \mu) \rangle_{s_0(\xi)} &= \frac{3}{16\pi \xi^2} \int x_1^3 (1 - \mu) F d^2 \omega_1 \\ &= \frac{3}{16\pi x \xi} \int \xi_1 (1 - \mu_0) F d\xi_1 d\phi_0 \\ &= \frac{3}{8 x \xi^2} \int_{\xi/(1+2\xi)}^{\xi} (\xi - \xi_1) F d\xi_1 = \frac{\xi}{x} [s_0(\xi) - s_1(\xi)] = \frac{\xi^2}{x} S_1(\xi), \end{aligned} \quad (73)$$

where we have used the invariant given by Equation (4) and changed the variables according to Equation (39). Thus, for the fixed electron and photon energies and the angle between their momenta, the mean momentum transmitted to the electron gas in the direction of the initial photon propagation $\boldsymbol{\omega}$, in accordance with (48), (46) and (73), is (NP94)

$$\langle Q_3 \rangle_{s_0(\xi)} = \langle x - x_1 \mu \rangle_{s_0(\xi)} = \left(x + x \xi - \gamma \xi + \frac{\xi^2}{x} \right) S_1(\xi). \quad (74)$$

In contrast to NP94, we are now interested to know the total momentum transfer. Obviously, by symmetry, it has to lie in the $(\boldsymbol{\Omega}, \boldsymbol{\omega})$ plane. Averaging expression (68) for Q_1 over angles is not easy, but we can compute the momentum transferred along the electron momentum: $Q_\Omega \equiv x\zeta - x_1\zeta_1$. Using identity $x_1\zeta_1 = (\gamma x_1 - \xi_1)/p$ and substituting Eqs. (41) and (42), similarly to Equation (40), we get

$$\begin{aligned} \langle x_1 \zeta_1 \rangle_{s_0(\xi)} &= \frac{3}{16\pi} \frac{1}{\xi^2} \int_{\xi/(1+2\xi)}^{\xi} F d\xi_1 \int_0^{2\pi} x_1 \zeta_1 d\phi_0 \\ &= \frac{3}{8\xi^2} \int_{\xi/(1+2\xi)}^{\xi} F d\xi_1 \frac{1}{p} \left[\left(\frac{\gamma x}{\xi} + \frac{\gamma x}{\xi^2} - \frac{\gamma^2}{\xi} - 1 \right) \xi_1 + \gamma \left(\gamma - \frac{x}{\xi} \right) \right] \\ &= \frac{1}{p} \left(\gamma x + \frac{\gamma x}{\xi} - \gamma^2 - \xi \right) s_1(\xi) + \frac{\gamma}{p} \left(\gamma - \frac{x}{\xi} \right) s_0(\xi) \\ &= \frac{1}{p} \left[\gamma^2 \xi S_1(\xi) + \gamma x \xi S_2(\xi) - \xi s_1(\xi) \right], \end{aligned} \quad (75)$$

and using identity $x\zeta = (\gamma x - \xi)/p$, we finally obtain

$$\langle Q_\Omega \rangle_{s_0(\xi)} = \langle x\zeta - x_1\zeta_1 \rangle_{s_0(\xi)} = \frac{1}{p} (\gamma x + \gamma x \xi - \gamma^2 \xi - \xi^2) S_1(\xi). \quad (76)$$

The momentum along $\mathbf{e}_1(\boldsymbol{\omega}, \boldsymbol{\Omega})$ is then simply

$$\begin{aligned} \langle Q_1 \rangle_{s_0(\xi)} &= \frac{\langle Q_\Omega \rangle - \zeta \langle Q_3 \rangle}{\sqrt{1 - \zeta^2}} s_0(\xi) \\ &= \frac{S_1(\xi)}{p \sqrt{1 - \zeta^2}} \left(\xi - 2\gamma \frac{\xi^2}{x} + \frac{\xi^3}{x^2} \right) = -p \sqrt{1 - \zeta^2} \xi S_1(\xi). \end{aligned} \quad (77)$$

Thus the total transferred momentum \mathbf{Q} averaged over directions of scattered photons can be decomposed into two components along basis vectors:

$$\langle \mathbf{Q} \rangle = \langle Q_1 \rangle \mathbf{e}_1(\boldsymbol{\omega}, \boldsymbol{\Omega}) + \langle Q_3 \rangle \mathbf{e}_3. \quad (78)$$

3.4.2. Averaging over electron directions

As in previous sections, we choose to measure azimuth of the electron momentum Φ in the frame defined by Eqs. (65) from the projection of vector \mathbf{l}_3 onto the plane perpendicular to $\boldsymbol{\omega}$. Therefore,

$$\langle \mathbf{Q} \rangle = \langle Q_1 \rangle [\cos \Phi \mathbf{e}_1(\boldsymbol{\omega}, \mathbf{l}_3) + \sin \Phi \mathbf{e}_2(\boldsymbol{\omega}, \mathbf{l}_3)] + \langle Q_3 \rangle \mathbf{e}_3. \quad (79)$$

The total momentum transfer averaged over the electron distribution is obtained from Equation (69):

$$\bar{\mathbf{Q}} \bar{s}_0(x, \eta) = \frac{1}{x} \int_1^\infty p d\gamma \int_{-1}^1 d\zeta \xi \int_0^{2\pi} d\Phi \langle \mathbf{Q} \rangle_{s_0(\xi)} f_e(\gamma, \eta_e). \quad (80)$$

Obviously, the component along $\mathbf{e}_2(\boldsymbol{\omega}, \mathbf{l}_3)$ becomes zero, as f_e is an even function of Φ . The term along $\mathbf{e}_1(\boldsymbol{\omega}, \mathbf{l}_3)$ involves integration of f_e over azimuth with the weight $\cos \Phi$, and its averaged value is

$$\overline{f_e \cos \Phi} = \sum_{k=1}^2 \frac{f_k}{k(k+1)} P_k^1(\eta) P_k^1(\zeta), \quad (81)$$

where P_k^1 are the associated Legendre functions:

$$P_1^1(u) = \sqrt{1 - u^2}, \quad P_2^1(u) = 3u \sqrt{1 - u^2}. \quad (82)$$

It is worth mentioning that the isotropic component of the electron distribution f_0 does not contribute to the momentum transferred perpendicular to $\boldsymbol{\omega}$ by symmetry. Substituting expression (77) to Equation (80), we get the first component of vector

$$\bar{Q}_1 \bar{s}_0(x, \eta) = 4\pi x \sum_{k=1}^2 P_k^1(\eta) \int_1^\infty p \gamma d\gamma f_k \Delta_{1k}^\perp(x, \gamma), \quad (83)$$

where

$$\Delta_{1k}^\perp(x, \gamma) = \frac{1}{k(k+1)} \frac{1}{2\gamma x^2} \int_{-1}^1 P_k^1(\zeta) \xi \langle Q_1 \rangle_{s_0(\xi)} d\zeta. \quad (84)$$

Changing the integration variable to ξ , and introducing a set

of functions

$$\begin{aligned}\chi_{1n}^\perp(x, \gamma) &= \frac{1}{2\gamma p} \frac{1}{x^{3+n}} \int_{x(\gamma-p)}^{x(\gamma+p)} \sqrt{1-\xi^2} \langle Q_1 \rangle s_0(\xi) \xi^{n+1} d\xi \\ &= \frac{1}{2\gamma p} \frac{1}{p} \left[\frac{u^{3+n}}{3+n} \Psi_{2+n,1}(xu) - 2\gamma \frac{u^{4+n}}{4+n} \Psi_{3+n,1}(xu) \right. \\ &\quad \left. + \frac{u^{5+n}}{5+n} \Psi_{4+n,1}(xu) \right] \Bigg|_{u=\gamma-p}^{u=\gamma+p}, \quad n = 0, 1, \quad (85)\end{aligned}$$

we get

$$\Delta_{11}^\perp = \frac{1}{2} \chi_{10}^\perp, \quad \Delta_{12}^\perp = \frac{1}{2p} (\gamma \chi_{10}^\perp - \chi_{11}^\perp). \quad (86)$$

Now let us evaluate the component of vector (80) along \mathbf{e}_3 . Because $\langle Q_3 \rangle$ does not depend on azimuth Φ , the azimuthal integration just gives the averaged electron distribution given by Equation (25). Thus we get

$$\overline{Q}_3 \overline{s}_0(x, \eta) = 4\pi x \sum_{k=0}^2 P_k(\eta) \int_1^\infty p\gamma d\gamma f_k (\Delta_{0k} - \Delta_{1k} + \Delta_{1k}^*), \quad (87)$$

where

$$\Delta_{1k}^*(x, \gamma) = \frac{1}{2\gamma x^2} \int_{-1}^1 P_k(\zeta) \xi \langle x_1(1-\mu) \rangle s_0(\xi) d\zeta = \sum_{n=0}^k b_{nk} \chi_{1n}^*, \quad (88)$$

and

$$\begin{aligned}\chi_{1n}^*(x, \gamma) &= \frac{1}{2\gamma p} \frac{1}{x^{3+n}} \int_{x(\gamma-p)}^{x(\gamma+p)} \langle x_1(1-\mu) \rangle s_0(\xi) \xi^{n+1} d\xi \\ &= \frac{1}{2\gamma p} \frac{u^{4+n}}{4+n} \Psi_{3+n,1}(xu) \Bigg|_{u=\gamma-p}^{u=\gamma+p}, \quad n = 0, 1, 2. \quad (89)\end{aligned}$$

For isotropic electron distribution, the only function of interest is Δ_{10}^* which coincides with function $\Psi_1^*(x, \gamma)$ introduced by NP94. Now combining Eqs. (83) and (87), we get the momentum transfer along the symmetry axis of the electron distribution l_3 and perpendicular to it:

$$\overline{Q}_\parallel = \sqrt{1-\eta^2} \overline{Q}_1 + \eta \overline{Q}_3, \quad (90)$$

$$\overline{Q}_\perp = -\eta \overline{Q}_1 + \sqrt{1-\eta^2} \overline{Q}_3. \quad (91)$$

Expressions (87), (83) and (90) give the momentum transferred to the electron gas (in terms of one integral over the electron energy) along ω , perpendicular to that direction as well as along vector l_3 and perpendicular to it.

Similarly to Equation (62), we can also get the two components of the momentum transfer rate per unit volume:

$$\dot{P}_{1,3} = \frac{N_e \sigma_T}{c} \int \frac{dx}{x} \int d^2\omega I(x, \eta) \overline{Q}_{1,3} \overline{s}_0(x, \eta). \quad (92)$$

For mono-energetic electron distribution of Lorentz factor γ given by Equation (10), the momenta transferred along ω

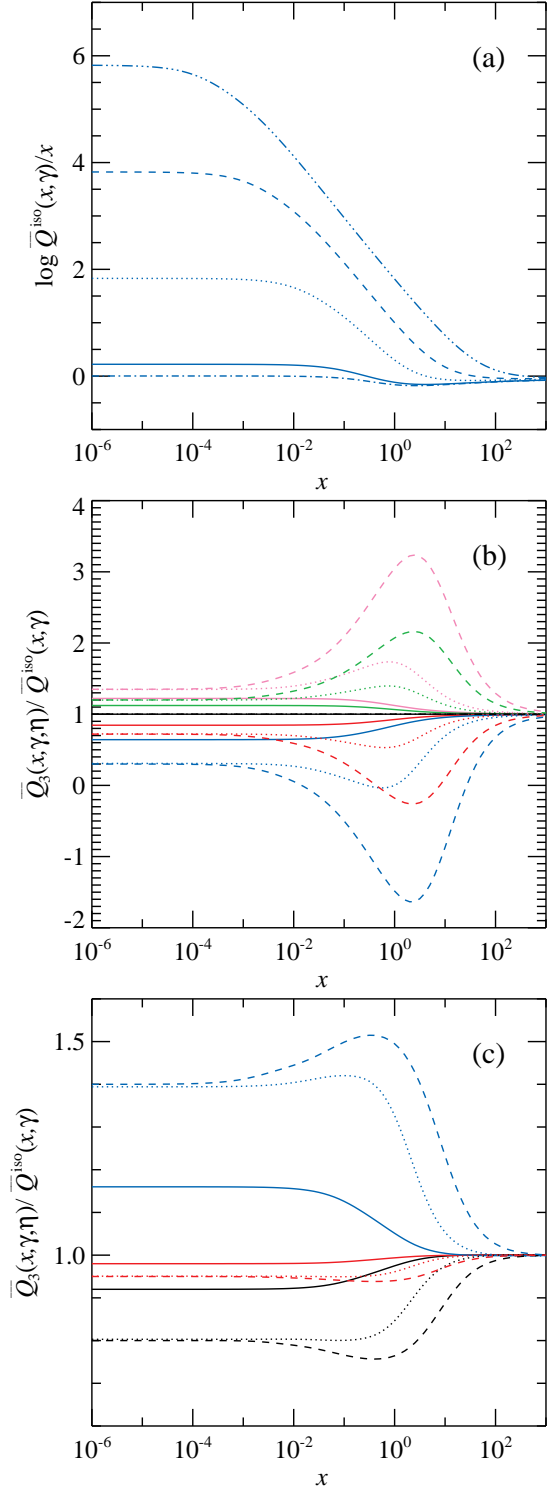


FIG. 5.— (a) Average momentum transferred to the electron gas per scattering $\overline{Q}_{\text{iso}}$ (in units of x) for isotropic mono-energetic electrons (with $f_1 = f_2 = 0$). Curves from bottom to top correspond to the electron momenta $p = 0.1, 1, 10, 10^2, 10^3$. The asymptotic value at small x in Thomson approximation is $1 + 2p^2/3$ [NP94; see Equation (95)]. (b) The momentum transferred along ω for anisotropic electrons with $f_1/f_0 = 1$ in units of the isotropic quantity $\overline{Q}_{\text{iso}}$. Solid, dotted and dashed curves correspond to $p = 1, 10, 100$, respectively. The curves from top to bottom correspond to $\eta = -1, -0.5, 0, 0.5, 1$ (pink, green, black, red, and blue curves, respectively). (c) Same as (b), but for the electron distribution (8) with the quadrupole term with $f_2/f_0 = 1$ for the same p and η as in panel (b). These are even functions of η , the curves from the bottom to the top correspond to $\eta = 0, 0.5, 1$. The flat parts of the curves correspond to the Thomson limit given by Equation (95).

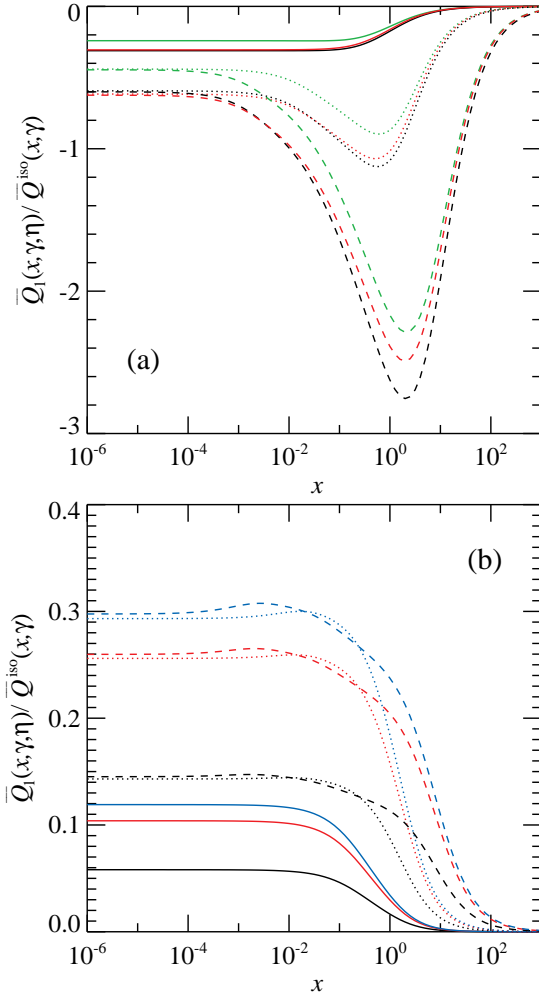


FIG. 6.— (a) Momentum transferred in the direction perpendicular to ω (in units of the momentum along ω for isotropic distribution) arising from the linear term in the electron distribution (8) with $f_1/f_0 = 1$. For $\eta = \pm 1$, the momentum is zero by symmetry. From the top to the bottom curves correspond to $\eta = -0.5$ (green), 0.5 (red), 0 (black). Solid, dotted and dashed curves correspond to $p = 1, 10, 100$, respectively. (b) Momentum transferred in the direction perpendicular to ω arising from the quadrupole term in the electron distribution (8) with $f_2/f_0 = 1$ for the same p as in panel (a). The curves from bottom to top correspond to $\eta = \pm 0.25, 0.5, 0.75$ (black, red, blue curves). The momentum is zero for $\eta = -1, 0, 1$ because of the symmetry. The flat parts of the curves correspond to the Thomson limit given by Equation (96).

and perpendicular to it are

$$\bar{Q}_3 \bar{s}_0(x, \gamma, \eta) = x \sum_{k=0}^2 \frac{f_k}{f_0} P_k(\eta) (\Delta_{0k} - \Delta_{1k} + \Delta_{1k}^*), \quad (93)$$

$$\bar{Q}_1 \bar{s}_0(x, \gamma, \eta) = x \sum_{k=1}^2 \frac{f_k}{f_0} P_k^1(\eta) \Delta_{1k}^\perp, \quad (94)$$

where we kept the notations for the functions \bar{Q}_3 and \bar{Q}_1 , but added the argument γ . To get the average momentum transferred in a single scattering act, one needs to divide these expression by the total cross-section $\bar{s}_0(x, \gamma, \eta)$. The ω component of the transferred momentum for isotropic electrons is shown in Figure 5a. As shown by NP94, the low-energy (Thomson) limit is given by $x(1 + 2p^2/3)$. The angular dependent corrections arising due to the dipole and quadrupole term in the electron distribution are shown in Figures 5b and

5c, respectively. While the component perpendicular to ω is zero for isotropic electrons, a substantial momentum component arises in the anisotropic case. For a large linear term of the electron distribution with $f_1/f_0 = 1$, the momentum transferred in that direction is shown in Figure 6a. Similar results in case of the quadrupole term with $f_2/f_0 = 1$, are shown in Figure 6b. In the Thomson limit, we get (see Appendix D)

$$\bar{Q}_3 \bar{s}_0(x, \gamma, \eta) = x \left[1 + \frac{2}{3} p^2 - \frac{f_1}{f_0} \eta \beta \frac{2}{15} (4\gamma^2 + 1) + \frac{f_2}{f_0} P_2(\eta) \frac{4}{15} p^2 \right], \quad (95)$$

$$\bar{Q}_1 \bar{s}_0(x, \gamma, \eta) = x \gamma^2 \beta \sqrt{1 - \eta^2} \times \left[-\frac{f_1}{f_0} \frac{1}{3} \left(1 + \frac{\beta^2}{5} \right) + \frac{2}{5} \frac{f_2}{f_0} \eta \beta \right]. \quad (96)$$

4. REDISTRIBUTION FUNCTIONS FOR ANISOTROPIC ELECTRONS

We would like to reduce the expression for the redistribution function (6) to a form suitable for calculations. For the electron distribution of the form (8), this function should depend on the energies of incoming and scattered photons x_1 and x , the corresponding (cosines of) polar angles η_1 and η as well as the difference in azimuth $\phi - \phi_1$ (or cosine of the scattering angle μ).

4.1. Integration over electron directions

The three-dimensional integral over \mathbf{p} in Equation (6) disappears due to the δ -function. For further simplifications we can also use the identity

$$\delta(\gamma_1 + x_1 - \gamma - x) = \gamma \delta(\underline{x} \cdot (\underline{p}_1 + \underline{x}_1) - \underline{x}_1 \cdot \underline{p}_1). \quad (97)$$

At this stage, we drop subscript 1 with the electron quantities and get

$$R(\mathbf{x}_1 \rightarrow \mathbf{x}) = \frac{3}{16\pi} \int \frac{d^3p}{\gamma} \delta(\Gamma) f_e(\gamma, \eta_e) F. \quad (98)$$

where

$$\Gamma = \gamma(x_1 - x) - p(x_1 \omega_1 - x \omega) \cdot \mathbf{\Omega} - q. \quad (99)$$

The angular integrals in Equation (98) need the introduction of a suitable coordinate system. Often the polar axis is taken along the direction of the scattered photon ω (see e.g. Nagirner & Poutanen 1993, 1994). However, the easiest and the most transparent way, is to choose the polar axis along the direction of the transferred momentum as was proposed by Aharonian & Atoyan (1981) (see also Prasad et al. 1986)

$$\mathbf{n} \equiv (x_1 \omega_1 - x \omega) / Q, \quad (100)$$

where

$$Q^2 = (x_1 \omega_1 - x \omega)^2 = x^2 + x_1^2 - 2x x_1 \mu = (x - x_1)^2 + 2q. \quad (101)$$

With this definition we get:

$$\cos \kappa \equiv \mathbf{n} \cdot \omega = (x_1 \mu - x) / Q, \quad \sin \kappa = x_1 \sqrt{1 - \mu^2} / Q \quad (102)$$

and

$$\cos \alpha \equiv \mathbf{n} \cdot \mathbf{l}_3 = (x_1 \eta_1 - x \eta) / Q. \quad (103)$$

Thus one of the integration variables becomes $\cos \theta = \mathbf{\Omega} \cdot \mathbf{n}$ and another is azimuth Φ . The redistribution function (98)

then can be written as

$$R(\mathbf{x}_1 \rightarrow \mathbf{x}) = \frac{3}{16\pi} \int_1^\infty p d\gamma \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\Phi f_e(\gamma, \eta_e) F \delta(\Gamma), \quad (104)$$

where now

$$\Gamma = \gamma(x_1 - x) - q - pQ \cos\theta. \quad (105)$$

Integrating first over $\cos\theta$ using the δ -function we get

$$R(\mathbf{x}_1 \rightarrow \mathbf{x}) = \frac{3}{16\pi} \int_{\gamma_*}^\infty d\gamma \frac{1}{Q} \int_0^{2\pi} d\Phi f_e(\gamma, \eta_e) F, \quad (106)$$

where we need to substitute

$$\cos\theta = \frac{\gamma(x_1 - x) - q}{pQ} \quad (107)$$

to the expressions for η_e and F (see below). This yields

$$\sin\theta = \frac{b}{\sqrt{r} pQ}, \quad (108)$$

where

$$b = \sqrt{r} \sqrt{p^2 Q^2 - [\gamma(x_1 - x) - q]^2}, \quad r = \frac{1 + \mu}{1 - \mu}. \quad (109)$$

The lower limit for the integral over γ comes from the requirement that $|\cos\theta| \leq 1$:

$$\gamma \geq \gamma_*(x, x_1, \mu) = \frac{1}{2} (x - x_1 + Q \sqrt{1 + 2/q}). \quad (110)$$

4.2. Integration over the azimuth

In order to calculate the azimuthal integral in Equation (106) we have to express ξ and ξ_1 (that enter the expression for F) and η_e in terms of the integration variable Φ . We measure the azimuth Φ from the projection of $\boldsymbol{\omega}$ onto the plane normal to \mathbf{n} , so that in this system

$$\boldsymbol{\omega} = (\sin\kappa, 0, \cos\kappa) \quad (111)$$

and the unit vector along the electron momentum is

$$\boldsymbol{\Omega} = (\sin\theta \cos\Phi, \sin\theta \sin\Phi, \cos\theta). \quad (112)$$

Thus we can express the angle between the electron momentum and \mathbf{l}_3 (see Fig. 1) through Φ :

$$\eta_e \equiv \boldsymbol{\Omega} \cdot \mathbf{l}_3 = \cos\theta \cos\alpha + \sin\theta \sin\alpha \cos(\chi - \Phi), \quad (113)$$

where χ is the azimuth of the vector \mathbf{l}_3 in the \mathbf{n} frame. We can also write

$$\eta \equiv \boldsymbol{\omega} \cdot \mathbf{l}_3 = \cos\kappa \cos\alpha + \sin\kappa \sin\alpha \cos\chi, \quad (114)$$

and use this expression to obtain $\cos\chi$. Substituting it to Equation (113) we thus express the electron polar angle η_e in Equation (8) through the integration variable Φ .

The kernel F depends on the four-products ξ and ξ_1 , which can be rewritten as

$$\xi_1 = x(\gamma - p\zeta), \quad \xi = q + \xi_1, \quad (115)$$

where

$$\zeta \equiv \boldsymbol{\Omega} \cdot \boldsymbol{\omega} = \cos\theta \cos\kappa + \sin\theta \sin\kappa \cos\Phi. \quad (116)$$

Equation (115) then can be transformed to

$$\xi_1 = \frac{q}{Q^2} (d_- - b \cos\Phi), \quad \xi = \frac{q}{Q^2} (d_+ - b \cos\Phi), \quad (117)$$

where we defined

$$\begin{aligned} d_- &= \gamma(x + x_1) - x(x - x_1\mu), \\ d_+ &= \gamma(x + x_1) + x_1(x_1 - x\mu) = d_- + Q^2, \end{aligned} \quad (118)$$

which have the following property:

$$\begin{aligned} (d_-^2 - b^2)/Q^2 &= (\gamma - x)^2 + r \equiv a_-^2, \\ (d_+^2 - b^2)/Q^2 &= (\gamma + x_1)^2 + r \equiv a_+^2. \end{aligned} \quad (119)$$

Function F in the azimuthal integral in Equation (106) is an even function of Φ . Therefore the terms in f_e containing $\sin\Phi$ give zero contribution. Neglecting these terms we can express the azimuthal integral as

$$\int_0^{2\pi} \eta_e F d\Phi = \int_0^{2\pi} \overline{\eta_e} F d\Phi, \quad (120)$$

$$\int_0^{2\pi} \eta_e^2 F d\Phi = \int_0^{2\pi} \overline{\eta_e^2} F d\Phi, \quad (121)$$

where

$$\overline{\eta_e} = \cos\theta \cos\alpha + \sin\theta \sin\alpha \cos\chi \cos\Phi, \quad (122)$$

$$\begin{aligned} \overline{\eta_e^2} &= \cos^2\theta \cos^2\alpha + \sin^2\theta \sin^2\alpha \sin^2\chi \\ &\quad + 2 \sin\theta \sin\alpha \cos\theta \cos\alpha \cos\chi \cos\Phi \\ &\quad + \sin^2\theta \sin^2\alpha \cos 2\chi \cos^2\Phi. \end{aligned} \quad (123)$$

Thus the expansion (8) (with η_e and η_e^2 substituted by $\overline{\eta_e}$ and $\overline{\eta_e^2}$, respectively) is a quadratic function of $\cos\Phi$. Expressing

$$\cos\Phi = -\frac{Q^2}{2b} \frac{\xi + \xi_1}{q} + \frac{d_- + d_+}{2b}, \quad (124)$$

$$\cos^2\Phi = \frac{Q^4}{b^2} \frac{\xi\xi_1}{q^2} - \frac{Q^2(d_+ + d_-)}{2b^2} \frac{\xi + \xi_1}{q} + \frac{d_+^2 + d_-^2}{2b^2}$$

and using the identity $\xi = \xi_1 + q$, we obtain an expansion of f_e that is symmetric in ξ and ξ_1 :

$$\overline{f_e}(\gamma) = c_0 + c_\Sigma \frac{\xi + \xi_1}{q} + c_\Pi \frac{\xi\xi_1}{q^2}. \quad (125)$$

The coefficients c_0 , c_Σ and c_Π can be represented in the form:

$$\begin{aligned} c_0 &= f_0 + c_{01}f_1 + c_{02}f_2, \\ c_\Sigma &= c_{11}f_1 + c_{12}f_2, \\ c_\Pi &= c_{22}f_2, \end{aligned} \quad (126)$$

where the coefficients in front of $f_{0,1,2}$ can be easily derived after lengthy but straightforward calculation:

$$\begin{aligned} c_{01} &= \frac{2\rho - \epsilon + \epsilon_1}{2p(1 + \mu)}, \\ c_{11} &= -\frac{\epsilon + \epsilon_1}{2p(1 + \mu)}, \\ c_{02} &= \frac{3}{4p^2(1 + \mu)^2} [(\epsilon - \rho)^2 + (\epsilon_1 + \rho)^2 + \lambda(a_-^2 + a_+^2)] - \frac{1}{2}, \\ c_{12} &= -\frac{3}{4p^2(1 + \mu)^2} [(\epsilon + \epsilon_1)(2\rho - \epsilon + \epsilon_1) + \lambda(d_- + d_+)], \\ c_{22} &= \frac{3}{2p^2(1 + \mu)^2} [(\epsilon + \epsilon_1)^2 + \lambda Q^2]. \end{aligned} \quad (127)$$

Here we defined

$$\epsilon \equiv x(\eta_1 - \eta\mu), \quad \epsilon_1 \equiv x_1(\eta - \eta_1\mu), \quad \rho = \gamma(\eta + \eta_1), \\ \lambda = \mu^2 + \eta^2 + \eta_1^2 - 2\mu\eta\eta_1 - 1. \quad (128)$$

The redistribution function (106) is then expressed as

$$R(x_1, \omega_1 \rightarrow x, \omega) = \frac{3}{8} \int_{\gamma_*(x, x_1, \mu)}^{\infty} d\gamma [c_0 R_0 + c_\Sigma R_\Sigma + c_\Pi R_\Pi], \quad (129)$$

where we have introduced three functions

$$R_0(x, x_1, \mu, \gamma) = \frac{1}{\pi Q} \int_0^\pi F d\Phi, \quad (130)$$

$$R_\Sigma(x, x_1, \mu, \gamma) = \frac{1}{\pi Q q} \int_0^\pi (\xi + \xi_1) F d\Phi, \quad (131)$$

$$R_\Pi(x, x_1, \mu, \gamma) = \frac{1}{\pi Q q^2} \int_0^\pi \xi \xi_1 F d\Phi. \quad (132)$$

Alternatively, we can represent the redistribution function as a sum of three terms arising from the corresponding three terms in the electron distribution:

$$R(x_1, \omega_1 \rightarrow x, \omega) = \frac{3}{8} \int_{\gamma_*(x, x_1, \mu)}^{\infty} d\gamma [f_0 R_0 + f_1 R_1 + f_2 R_2], \quad (133)$$

where

$$R_1(x, \eta; x_1, \eta_1; \mu; \gamma) = c_{01} R_0 + c_{11} R_\Sigma, \quad (134)$$

$$R_2(x, \eta; x_1, \eta_1; \mu; \gamma) = c_{02} R_0 + c_{12} R_\Sigma + c_{22} R_\Pi.$$

Using the Klein-Nishina cross-section (2) in the form

$$F = 2 + \frac{q^2 - 2q - 2}{q} \left(\frac{1}{\xi_1} - \frac{1}{\xi} \right) + \frac{1}{\xi^2} + \frac{1}{\xi_1^2}, \quad (135)$$

(and remembering that $\xi = \xi_1 + q$), we see that the integrals (130)–(132) involve integrals of types

$$\int_0^\pi \xi^s d\Phi, \quad \int_0^\pi \xi_1^s d\Phi, \quad (136)$$

where $s = -2, \dots, 2$. The integrals over non-negative powers of ξ and ξ_1 are trivial. For the negative powers, using Equations (117) and (119) we get (see Nagirner & Poutanen 1993, for details):

$$\int_0^\pi \frac{d\Phi}{\xi_1} = \frac{\pi Q}{q} \frac{1}{a_-}, \quad \int_0^\pi \frac{d\Phi}{\xi_1^2} = \frac{\pi Q}{q^2} \frac{d_-}{a_-^3} \quad (137)$$

and similar equations for ξ which we get by substituting ξ , a_+ and d_+ for ξ_1 , a_- and d_- , respectively.

After some straightforward algebra we get the expressions for R_0 , R_Σ and R_Π :

$$R_0 = \frac{2}{Q} + \frac{q^2 - 2q - 2}{q^2} \left(\frac{1}{a_-} - \frac{1}{a_+} \right) + \frac{1}{q^2} \left(\frac{d_-}{a_-^3} + \frac{d_+}{a_+^3} \right), \quad (138)$$

which was obtained by Aharonian & Atoyan (1981) (see also Nagirner & Poutanen 1993),

$$R_\Sigma = \frac{2}{Q^3} (d_- + d_+) + \left(1 - \frac{2}{q} \right) \left(\frac{1}{a_-} + \frac{1}{a_+} \right) + \frac{1}{q^2} \left(\frac{d_-}{a_-^3} - \frac{d_+}{a_+^3} \right) \quad (139)$$

and

$$R_\Pi = \frac{2}{Q^5} \left(d_- d_+ + \frac{b^2}{2} \right) + \left(1 - \frac{2}{q} \right) \frac{1}{Q} + \frac{1}{q^2} \left(\frac{1}{a_-} - \frac{1}{a_+} \right). \quad (140)$$

Equation (129) (or alternatively equations [133] and [134]) together with our computed redistribution functions (138)–(140) and the coefficients (126)–(128) give the full analytical solution for the redistribution function describing scattering of arbitrary photons from the electron gas which anisotropy can be described by Equation (8).

4.3. Alternative redistribution functions

An alternative expression for the redistribution function $R(x_1 \rightarrow x)$ can be obtained if we compute the moments

$$R_\phi(x, x_1, \mu, \gamma) = \frac{1}{\pi Q} \int_0^\pi \cos \Phi F d\Phi \\ = \frac{q^2 - 2q - 2}{q^2} \frac{1}{b} \left(\frac{d_-}{a_-} - \frac{d_+}{a_+} \right) + \frac{b}{q^2} \left(\frac{1}{a_-^3} + \frac{1}{a_+^3} \right), \quad (141)$$

$$R_{\phi\phi}(x, x_1, \mu, \gamma) = \frac{1}{\pi Q} \int_0^\pi \cos^2 \Phi F d\Phi \\ = \frac{1}{Q} + \frac{Q^3}{b^2} \left(1 - \frac{2}{q} \right) + \frac{q^2 - 2q - 2}{q^2} \frac{1}{b^2} \left(\frac{d_-^2}{a_-^3} - \frac{d_+^2}{a_+^3} \right) \\ - \frac{Q^2}{q^2 b^2} \left(\frac{d_-}{a_-} + \frac{d_+}{a_+} \right) + \frac{1}{q^2} \left(\frac{d_-}{a_-^3} + \frac{d_+}{a_+^3} \right). \quad (142)$$

The expressions for R_1 and R_2 then take the form:

$$R_1 = d_{01} R_0 + d_{11} R_\phi, \quad (143)$$

$$R_2 = d_{02} R_0 + d_{12} R_\phi + d_{22} R_{\phi\phi},$$

where

$$d_{01} = \cos \theta \cos \alpha,$$

$$d_{11} = \sin \theta \sin \alpha \cos \chi = \frac{b}{p(1 + \mu)Q^2} (\epsilon + \epsilon_1),$$

$$d_{02} = \frac{3}{2} (\cos^2 \theta \cos^2 \alpha + \sin^2 \theta \sin^2 \alpha \sin^2 \chi) - \frac{1}{2} \\ = \frac{3}{2} (2d_{01}^2 - d_{11}^2 + \sin^2 \theta - \cos^2 \alpha) - \frac{1}{2}, \quad (144)$$

$$d_{12} = 3 \cos \theta \cos \alpha \sin \theta \sin \alpha \cos \chi = 3 d_{01} d_{11},$$

$$d_{22} = \frac{3}{2} \sin^2 \theta \sin^2 \alpha \cos 2\chi \\ = \frac{3}{2} (2d_{11}^2 - d_{01}^2 - \sin^2 \theta + \cos^2 \alpha).$$

4.4. Approximate redistribution functions

Approximate forms of Equations (138)–(140) can be obtained by making certain simplifying assumptions about the scattering. For example, in the Thomson regime in the electron rest frame the Klein-Nishina kernel F is just $1 + \mu_0^2$. Assuming further isotropic scattering in that frame and substituting F by $4/3$, we now get for the integrals (130)–(132):

$$R_0 \approx \frac{4}{3Q}, \quad (145)$$

$$R_\Sigma \approx \frac{4}{3Q^3} (d_- + d_+), \quad (146)$$

$$R_\Pi \approx \frac{4}{3Q^5} \left(d_- d_+ + \frac{b^2}{2} \right). \quad (147)$$

The expression for R_0 was derived by Arutyunyan & Nikogosyan (1980). For the alternative

functions (141), (142), we then have

$$R_\phi = 0, \quad R_{\phi\phi} = \frac{2}{3Q}. \quad (148)$$

These then give

$$R_1 \approx d_{01}R_0 = \cos\theta \cos\alpha R_0, \quad (149)$$

$$R_2 \approx \left(d_{02} + \frac{1}{2}d_{22}\right) R_0 = P_2(\cos\theta) P_2(\cos\alpha) R_0, \quad (150)$$

with $\cos\theta$ and $\cos\alpha$ given by Equations (107) and (103), respectively. The approximate expressions are better than 50 per cent accurate in the Thomson regime for $x_1\gamma < 0.1$ at all scattered photon energies.

4.5. Relation to the mean powers of photon energies

The relation between the redistribution function averaged over any electron distribution and the mean powers of photon energies follows directly from their definitions (6), (7) and (36):

$$\overline{x_1^j} \overline{s_0}(x, \eta) = \frac{1}{x} \int x_1^{j+1} dx_1 \int d^2\omega_1 R(x \rightarrow x_1). \quad (151)$$

This relation is valid for any electron distribution. Comparing Equations (133) and (49), we get a relation between the functions depending on the electron energy:

$$\Delta_{jk}(x, \gamma) P_k(\eta) = \frac{3}{32\pi\gamma p x^{j+1}} \times \int_{x^-(x, \gamma)}^{x_m(x, \gamma)} x_1^{j+1} dx_1 \int R_k(x_1, \eta_1; x, \eta; \mu; \gamma) d^2\omega_1, \quad (152)$$

where $\eta_1 = \eta\mu + \sqrt{1-\eta^2}\sqrt{1-\mu^2}\cos\Phi$ and R_0 depends only on the scattering angle μ , but not η, η_1 . The integrals over the solid angle can be represented as the integrals over $d\mu$ and $d\Phi$, where $\Phi \in [0, 2\pi]$ and the limits on $\mu, \mu_m(x_1, x, \gamma)$ and $\mu^+(x_1, x, \gamma)$, are given by Equations (F6)–(F8) with the arguments x and x_1 reversed. Using Equations (87) and (84), we also get

$$\Delta_{1k}^*(x, \gamma) P_k(\eta) = \frac{3}{32\pi\gamma p x^2} \int x_1^2 dx_1 \int (1-\mu) R_k d^2\omega_1, \quad (153)$$

$$\Delta_{1k}^\perp(x, \gamma) P_k^1(\eta) = \frac{3}{32\pi\gamma p x^2} \int x_1^2 dx_1 \int \sqrt{1-\mu^2} \cos\Phi R_k d^2\omega_1. \quad (154)$$

In order to check the accuracy of our derivations we compared the left hand sides of Equations (152)–(154) to the right hand sides, where the integrals were performed numerically and obtained consistent results.

5. APPLICATIONS

5.1. Examples of redistribution functions

Now we demonstrate the properties of the derived redistribution functions. We consider a volume filled by electrons with the angular distribution given by Equation (8). The emissivity in a direction ω at energy x can be obtained from the radiative transfer equation (5) and is given by the integral over the redistribution function

$$\epsilon(x) = \sigma_T N_e x^2 \int_0^\infty \frac{dx_1}{x_1^2} \int d^2\omega_1 I(x_1) R(x_1, \omega_1 \rightarrow x, \omega), \quad (155)$$

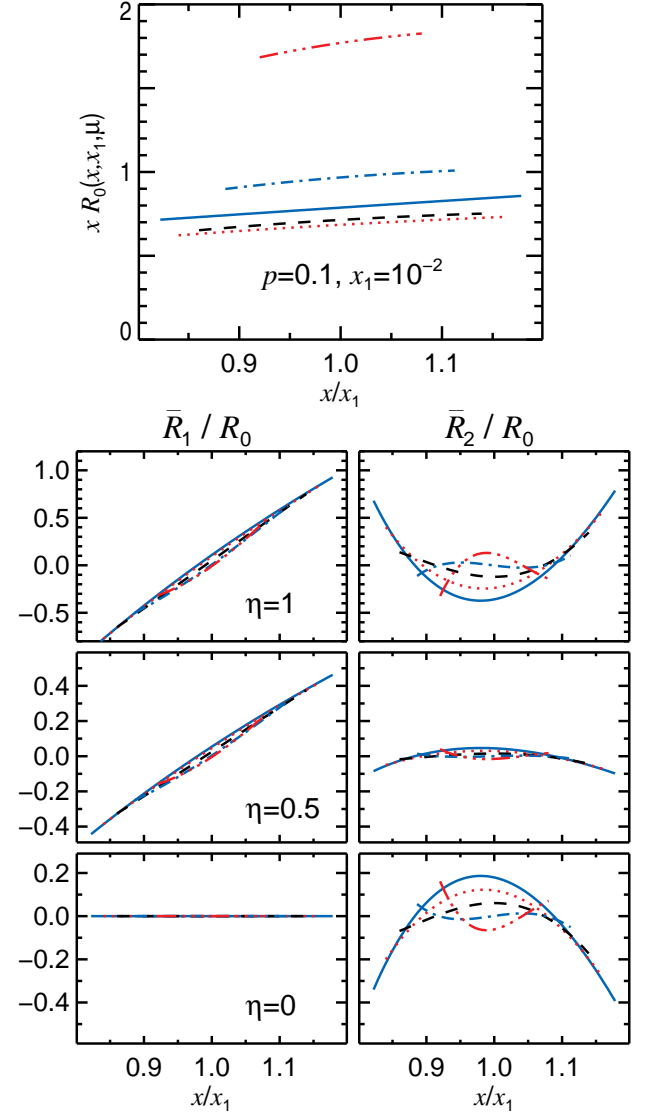


FIG. 7.— Redistribution functions for anisotropic electrons at a given scattering angle. The incident photon energy is $x_1 = 10^{-2}$ and the electron momentum $p = 0.1$. The upper panel shows the photon (number) emissivity for isotropic electrons. The solid, dotted, dashed, dot-dashed and dot-dot-dashed curves correspond to the cosine of scattering angle $\mu = -2/3, -1/3, 0, 1/3, 2/3$, respectively. The lower left panels show the ratio \bar{R}_1/R_0 , while the right panels show \bar{R}_2/R_0 as a function of the ratio of the scattered to the incident photon energies. The three rows of panels corresponds to the different observer directions $\eta = 1, 0.5, 0$.

where $I(x_1) = 2m_e(m_e c^2/h)^3 x_1^3 n(x_1)$ is the specific intensity of the incident radiation normalized to the photon density as

$$N_{ph} = \frac{1}{m_e c^3} \int d^2\omega \int I(x) \frac{dx}{x}. \quad (156)$$

Let us consider mono-energetic (with energy γ) electron distribution (10). Consider also a monochromatic source of isotropic seed photons at energy x_1 with total photon number density N_{ph} . According to Equation (133) we can write the emissivity at an observer direction η for a given scattering angle as

$$\bar{\epsilon}(x, \eta, \mu) = \frac{3}{32\pi} m_e c^3 \sigma_T N_e N_{ph} \frac{x^2}{x_1} \frac{1}{p\gamma} \left[R_0 + \frac{f_1}{f_0} \bar{R}_1 + \frac{f_2}{f_0} \bar{R}_2 \right], \quad (157)$$

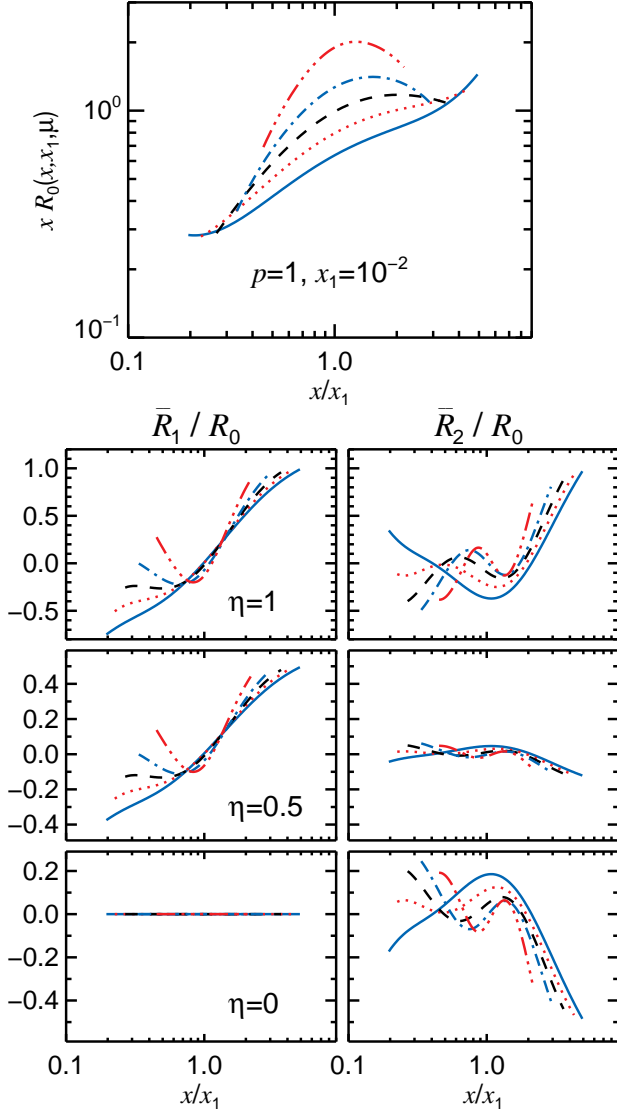


FIG. 8.— Same as Figure 7, but for $p = 1$. Note, that here the axes are in logarithmic units.

which is related to the scattering angle-averaged emissivity as $\epsilon(x) = \frac{1}{2} \int \bar{\epsilon}(x, \eta, \mu) d\mu$, and where (for $k = 1, 2$)

$$\bar{R}_k(x, \eta; x_1; \mu; \gamma) = \frac{1}{2\pi} \int_0^{2\pi} d\Phi R_k(x, \eta; x_1, \eta_1; \mu; \gamma) \quad (158)$$

and $\eta_1 = \eta\mu + \sqrt{1-\eta^2}\sqrt{1-\mu^2}\cos\Phi$. These functions obviously possess symmetry properties:

$$\bar{R}_1(x, -\eta; x_1; \mu; \gamma) = -\bar{R}_1(x, \eta; x_1; \mu; \gamma), \quad (159)$$

$$\bar{R}_2(x, -\eta; x_1; \mu; \gamma) = \bar{R}_2(x, \eta; x_1; \mu; \gamma). \quad (160)$$

We compute separately the emissivities resulting from three terms in the electron distribution, i.e. functions $R_0, \bar{R}_1, \bar{R}_2$ (see Equation [157]), and show in Figures 7–10 the function R_0 multiplied by x (i.e. quantity proportional to the photon number emissivity) for better visibility as well as the ratios \bar{R}_1/R_0 and \bar{R}_2/R_0 . The main behavior of the functions can be easily understood using formulae (149)–(150) derived in Thomson limit and isotropic scattering approximation. Averaging them over the azimuth and using relation $\overline{P_k(\eta_1)} = P_k(\eta)P_k(\mu)$, we

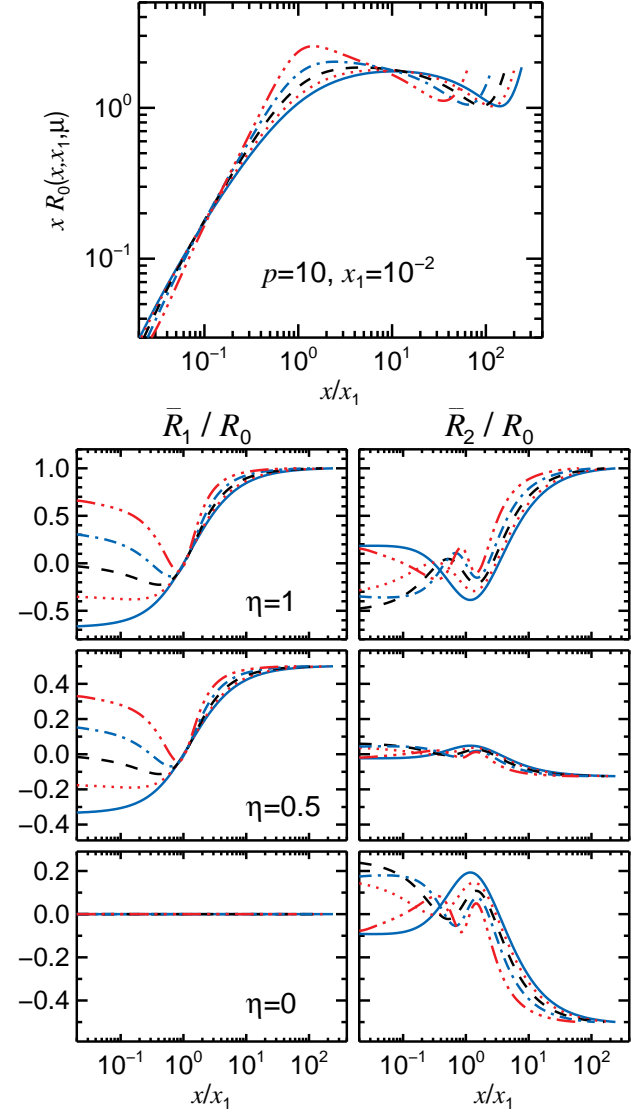


FIG. 9.— Same as Figure 8, but for $p = 10$.

get

$$\frac{\bar{R}_1}{R_0} \approx \frac{x_1\mu - x}{Q} \eta \cos\theta, \quad (161)$$

$$\frac{\bar{R}_2}{R_0} \approx \frac{x_1^2 P_2(\mu) - 2xx_1\mu + x^2}{Q^2} P_2(\eta) P_2(\cos\theta). \quad (162)$$

These approximate expressions become extremely accurate for high p (i.e. accuracy is about 10^{-3} at $p = 100$).

For a small electron momentum $p = 0.1$ and low photon energies $x_1 = 10^{-2}$, the exact redistribution functions are shown in Figure 7. In this regime, scattering is nearly coherent with the scattered photon energies bounded by (see Equation (F2)) $x^\pm/x_1 \approx 1 \pm p\sqrt{2(1-\mu)}$. In this regime, $|x-x_1|^2 \ll q \ll x, x_1$ and $\cos\theta \approx (x_1-x)/pQ$ is a nearly linear function of x/x_1 , because $Q/x_1 \approx \sqrt{2(1-\mu)}$. For μ not too close to 1, the azimuth averaging of $\cos\alpha$ gives $-\eta\sqrt{(1-\mu)/2}$ and thus $\bar{R}_1/R_0 \approx (1-x/x_1)\eta/(2p)$. For $\eta = 0$, the function is always zero, because of the symmetry. Similarly, the nearly quadratic dependence of \bar{R}_2/R_0 on energy results from

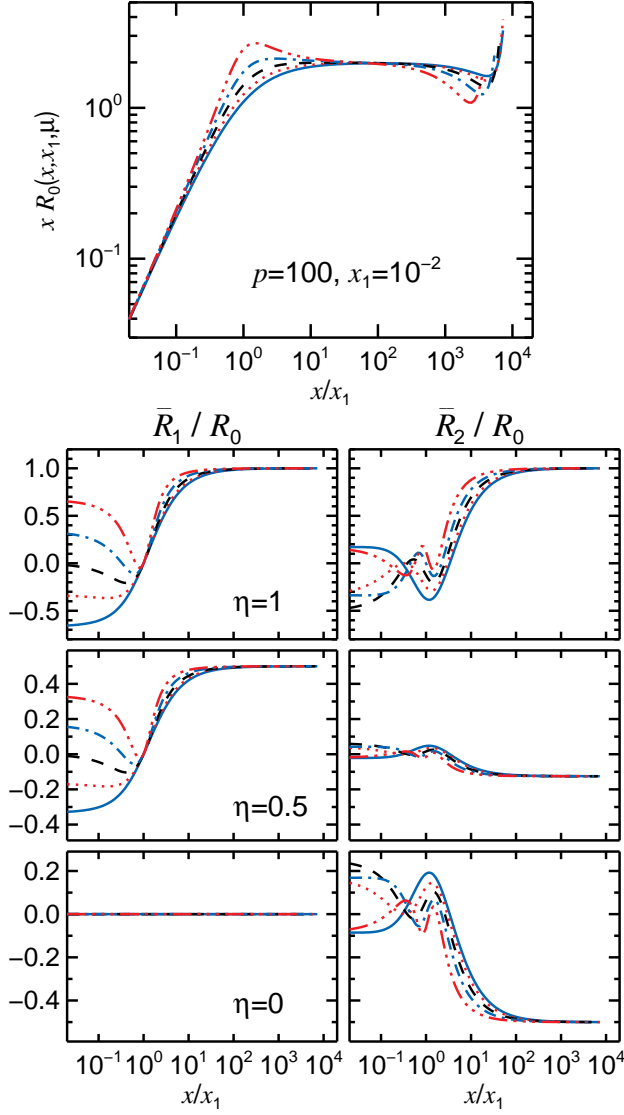


FIG. 10.— Same as Figure 8, but for $p = 100$.

the $\cos^2 \theta$ term, while at $\mu \approx 1/3$ the function becomes more complicated because of the cancellation in the $\overline{P}_2(\cos \alpha)$ term.

In the opposite limit of the relativistic electrons (see Figures 9 and 10), the approximation (145) for the function R_0 works fine up to $x_1 \gamma \lesssim 0.1$, while as said above the ratios \bar{R}_1/R_0 and \bar{R}_2/R_0 are very close to those given by Equations (161) and (162) for any photon and electron energies. At small scattered photon energies $x \ll x_1$, $xR_0 \propto x/x_1$, and

$$\bar{R}_1/R_0 \approx \eta \mu \cos \theta, \quad \bar{R}_2/R_0 \approx P_2(\eta)P_2(\mu)P_2(\cos \theta), \quad (163)$$

with $\cos \theta \approx 1 - x/x_1$. At high scattered photon energies $x \gg x_1$, the photons are scattered at large angles in the electron rest frame and therefore they are beamed in the direction of the incoming electrons. In that case, the angular distribution of the scattered photons resemble that of the electrons. In this regime $xR_0 \propto \text{const}$, and $R_1/R_0 \approx \eta$ and $R_2/R_0 \approx P_2(\eta)$, which gives the flat dependences clearly seen in Figures 9 and 10, and $\epsilon(x) \propto f_e(\gamma, \eta) x/x_1$.

5.2. Sunyaev–Zeldovich effect

Let consider a cloud of isotropic Maxwellian electrons of temperature $\Theta \equiv kT_e/m_e c^2$, which moves with velocity $c\beta_b$ (corresponding Lorentz factor Γ_b) through the isotropic cosmic microwave background of temperature $\Theta_{\text{cmb}} \equiv kT_{\text{cmb}}/m_e c^2$. We compute the thermal and kinematic Sunyaev–Zeldovich effects (Zeldovich & Sunyaev 1969; Sunyaev & Zeldovich 1972), i.e. the spectrum of the scattered radiation (and resulting deviations from the black body) as a function of Θ and the angle between the line of sight and the direction of motion.

One approach would be to make a Lorentz transformation of the incident radiation to the comoving frame, compute the Compton scattered radiation using the kernel corresponding to isotropic electron distribution, and then to Lorentz transform it back to the observer frame. Another way is to compute the electron distribution in the observer frame, approximate it by the expansion (8) and compute directly the Compton scattered radiation in the observer frame. The second approach might be favorable from numerical point of view if the object velocity is variable in space and/or time, as allows to pre-compute redistribution functions at a fixed grid of angles and photon energies.

5.2.1. Scattering in the comoving frame

Let us first compute the scattered radiation by a standard method considering scattering in the comoving frame. The relativistic Maxwellian distribution of electrons in the comoving frame (quantities with primes) is given by

$$f'_e(\mathbf{p}') = N'_e \frac{\exp(-\gamma'/\Theta)}{4\pi \Theta K_2(1/\Theta)}, \quad (164)$$

where K_2 is the modified Bessel function and N'_e is the electron density in that frame. The incident black body radiation occupation number is

$$n_{\text{bb}}(x) = \frac{1}{\exp(x_t) - 1}, \quad (165)$$

where $x_t = x/\Theta_{\text{cmb}} = hv/kT_{\text{cmb}}$. From the radiative transfer equation (5), in the limit of small optical depth, we get the correction to the black body spectrum:

$$\Delta n(x, \eta) = n(x, \eta) - n_{\text{bb}}(x) = -\tau_T \bar{s}_0(x') n_{\text{bb}}(x) + S(x, \eta), \quad (166)$$

where τ_T is the Lorentz invariant optical depth for Thomson scattering,

$$S(x, \eta) = \tau_T \frac{1}{x'} \int_0^\infty x'_1 dx'_1 \int d^2 \omega'_1 R^{\text{iso}}(\mathbf{x}'_1 \rightarrow \mathbf{x}') n_{\text{bb}}(x_1) \quad (167)$$

is the source function, and we used here the fact that the photon occupation number is Lorentz invariant. The energy transformation is given by Doppler shift $x = x' \mathcal{D}$ and $x_1 = x'_1 \mathcal{D}_1$ with the Doppler factors

$$\mathcal{D} = \frac{1}{\Gamma_b(1 - \beta_b \eta)}, \quad \mathcal{D}_1 = \Gamma_b(1 + \beta_b \eta'_1). \quad (168)$$

The relation between the angles is given by the aberration formula:

$$\eta' = \frac{\eta - \beta_b}{1 - \beta_b \eta}. \quad (169)$$

We note here that \bar{s}_0 is equal to unity with high accuracy, because scattering is in deep Thomson regime. The calculation

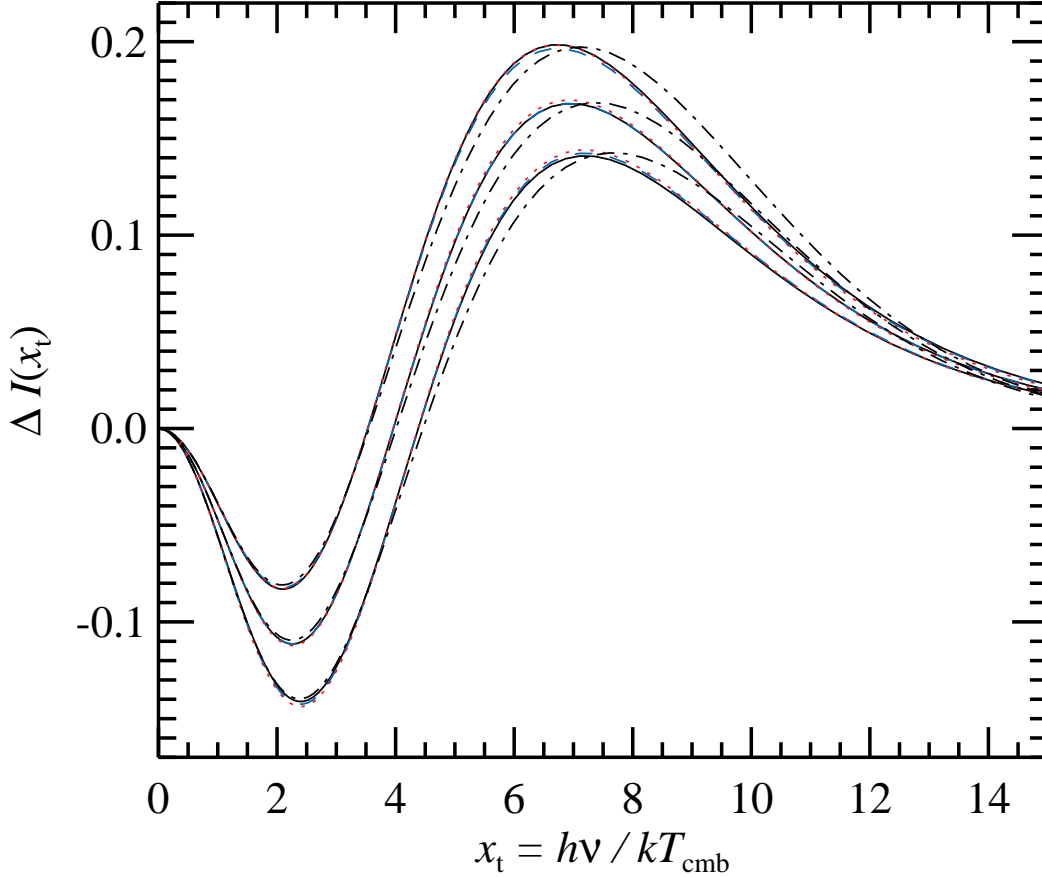


FIG. 11.— Deviation from the black body spectrum of cosmic microwave background radiation with $T_{\text{cmb}} = 2.7$ K resulting from the Compton scattering in a moving cloud of isotropic hot electrons (thermal and kinematic Sunyaev-Zeldovich effects). The electron temperature is $\Theta = 0.03$ and the cloud velocity $\beta_b = 0.01$. The solid curves are computed using Equations (166)–(167), considering scattering in the cloud frame, where electrons are isotropic. The dashed curves show the results using the formalism developed in this paper for anisotropic electrons, given by Equations (173)–(174). The dotted curves correspond to the semi-analytical approximation of the angle-averaged redistribution function given by Equations (175)–(176). The three different methods give nearly identical results. The three curves from bottom to the top correspond to the three viewing angles with $\eta = -0.98, 0, +0.98$. The dash-dotted curves show analytical approximation of Sazonov & Sunyaev (1998), which includes terms up to second order in β_b and Θ , as well as a cross-term $\beta_b\Theta$. It works reasonably well up to the temperatures $\Theta < 0.02$, but fails at higher temperatures in Wien tail.

of the redistribution function R^{iso} involves numerical integration over the Maxwellian distribution (see Equation [133]; note that $f_1 = f_2 = 0$) $f_0 = f'_e(\gamma) = f'_e(\mathbf{p})/N'_e$ given by Equation (164). Thus the source function (167) involves 4-dimensional integral to be taken numerically, which is rather time-consuming.

5.2.2. Scattering in the external frame

We can also compute the same effect directly in the external frame. The electron Lorentz factor in the comoving frame is related to the electron four-momentum in the external frame as

$$\gamma' = \Gamma_b(\gamma - p\beta_b\eta_e), \quad (170)$$

where η_e is the cosine of the angle between the electron momentum and the direction of cloud motion. Because the distribution function is Lorentz invariant, we easily get the electron distribution in the external frame:

$$\begin{aligned} f_e(\mathbf{p}) &= f'_e(\mathbf{p}') = N'_e \frac{\exp(-\gamma\Gamma_b/\Theta)}{4\pi\Theta K_2(1/\Theta)} \exp(p\Gamma_b\beta_b\eta_e/\Theta) \\ &\approx N'_e \frac{\exp(-\gamma/\Theta)}{4\pi\Theta K_2(1/\Theta)} \left[1 + \frac{\beta_b^2}{6}(p^2 - 3\gamma\Theta) + \beta_b p\eta_e + \frac{\beta_b^2 p^2}{3}P_2(\eta_e) \right], \end{aligned} \quad (171)$$

where $\beta_t = \beta_b/\Theta$ and we expanded the expression up to the second order in β_b . The electron density in that frame is:

$$N_e = \int f_e(\mathbf{p}) d^3\mathbf{p} = \Gamma_b N'_e. \quad (172)$$

The corresponding terms f_k of the electron distribution can be obtained from Equation (171) noting that $f_e(\gamma, \eta_e) = f_e(\mathbf{p})/\Gamma_b N'_e$. The change to the occupation number is:

$$\Delta n(x, \eta) = -\tau_T \bar{s}_0(x, \eta) n_{\text{bb}}(x) + S(x, \eta). \quad (173)$$

The scattering cross-section is given by Equation (26) and in Thomson limit is just $\bar{s}_0(x, \eta) \approx 1 - \beta_b\eta$. The source function is now

$$S(x, \eta) = \tau_T \frac{1}{x} \int_0^\infty x_1 n_{\text{bb}}(x_1) dx_1 \int d^2\omega_1 R(\mathbf{x}_1 \rightarrow \mathbf{x}), \quad (174)$$

where the redistribution function R given by Equation (133) is averaged over directions of incident photons, but still depends on the scattered photon direction η . This form of the source function is more favorable compared to Equation (167) from numerical point of view, as it can be tabulated in advance at a given grid of photon energies and angles. Computed directly it still involves numerical calculations of 4-dimensional integrals.

5.2.3. Isotropic scattering in Thomson regime in the electron rest frame

In Thomson limit (as in the case of Sunyaev-Zeldovich effect), the calculations in the external frame can be dramatically simplified. We can use the azimuthally averaged approximate expression (145), (161), and (162) for the redistribution functions:

$$\int d^2\omega_1 R(x_1 \rightarrow x) = 2\pi \int_{-1}^1 d\mu \frac{3}{8} \int_{\gamma_*(x, x_1, \mu)}^{\infty} d\gamma \left[f_0 R_0 + f_1 \bar{R}_1 + f_2 \bar{R}_2 \right], \quad (175)$$

Interestingly, R_0 does not depend on γ and in expressions for \bar{R}_1 and \bar{R}_2 it comes only through $\cos \theta \approx (x_1 - x)\gamma/Qp$ (because $q \ll x, x_1$, see eq. [107]). For the electron distribution given by Equation (171), the integrals over γ thus can be taken analytically:

$$\begin{aligned} \int d\gamma f_0 R_0 &= C \frac{1}{\Gamma_b} \left[1 - \frac{\beta_b^2}{6} \left(\frac{1}{\Theta^2} + 1 + \frac{\gamma_*}{\Theta} - \left(\frac{\gamma_*}{\Theta} \right)^2 \right) \right], \\ \int d\gamma f_1 R_1 &= C \beta_b \eta \frac{x_1 \mu - x}{Q} \frac{x_1 - x}{Q} \left(1 + \frac{\gamma_*}{\Theta} \right), \\ \int d\gamma f_2 R_2 &= C \frac{\beta_b^2}{3} P_2(\eta) \frac{x_1^2 P_2(\mu) - 2x x_1 \mu + x^2}{Q^2} \\ &\quad \times \left[P_2 \left(\frac{x_1 - x}{Q} \right) \left(2 + 2 \frac{\gamma_*}{\Theta} + \left(\frac{\gamma_*}{\Theta} \right)^2 \right) + \frac{1}{2\Theta^2} \right], \end{aligned} \quad (176)$$

where the proportionality coefficient $C = R_0 \exp(-\gamma_*/\Theta)/[4\pi K_2(1/\Theta)]$. The zeroth order term in β_b was derived by Poutanen (1994), see also Poutanen & Svensson (1996).

Evaluation of the source function (174) now involves only two numerical integrations over the photon energy x_1 and cosine of the scattering angle μ , reducing the computational time by 2-3 orders of magnitude.

For all three methods we numerically compute the correction function for the black body intensity

$$\Delta I(x) = \frac{1}{\tau_T} x_i^3 \Delta n(x, \eta), \quad (177)$$

and compare the results of calculations in Figure 11. The three different methods give nearly identical results.

6. CONCLUSIONS

We have developed the exact analytical theory of Compton scattering by anisotropic distribution of electrons that can be represented by a second order polynomial over cosine of some angle (dipole and quadrupole anisotropy). For the total

cross-section, we reduce the 9-dimensional integral to a single integral over the electron energy. Analogous expressions have been derived for the mean energy of the scattered photons and its dispersion. We also obtained analytical expressions for the radiation pressure force acting on the electron gas. These moments can be used for analytical estimations as well as for the numerical solutions of the kinetic equations in the Fokker-Planck approximation (see e.g. Vurm & Poutanen 2009).

Furthermore, the expression for the redistribution function describing angle-dependent Compton scattering by anisotropic electrons is reduced to a single integral over the electron energy. Exact analytical formulae valid for any photon and electron energy are derived in the case of monoenergetic electrons. We have also derived approximate expressions for the redistribution function, assuming isotropic scattering in the electron rest frame, which are very accurate in the case of relativistic electrons interacting with soft photons in Thomson regime.

We applied the developed formalism to the accurate calculations of the thermal and kinematic Sunyaev-Zeldovich effects for arbitrary electron distributions. A very similar problem arises in outflowing coronae around accreting black holes and neutron stars, where the bulk motion causes electron anisotropy. Another application could be a computation of the radiative transport in the synchrotron self-Compton sources with ordered magnetic field, where the electron distribution can have strong deviations from the isotropy because of pitch angle-dependent cooling. These problems will be considered in future publications.

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TABLE 1
COEFFICIENTS a_{jn} AND A_{jn} .

n	0	1	2
a_{0n}	1	1	13/10
a_{1n}	1	3/2	47/20
a_{2n}	1	2	15/4
A_{1n}	1	21/10	147/40
A_{2n}	6/5	53/20	159/35
A_{3n}	1	14/5	47/8
A_{4n}	7/5	22/5	341/35
A_{5n}	1/5	2	401/70
A_{6n}	1/10	1/5	207/280
A_{7n}	3/10	3/5	281/280

APPENDIX

A. FUNCTIONS S_j AND S_j

All function s_j and S_j can be expanded to the series, which converge in the region $\xi < 1/2$. It is easy to show that

$$s_j(\xi) = \sum_{n=0}^{\infty} a_{jn} (-2\xi)^n, \quad S_j(\xi) = \sum_{n=0}^{\infty} A_{jn} (-2\xi)^n, \quad (A1)$$

where

$$a_{0n} = \frac{3}{8} \left[n + 2 + \frac{2}{n+1} + \frac{8}{n+2} - \frac{16}{n+3} \right], \quad a_{1n} = \frac{1}{8} \left[n(n+5) + \frac{24}{n+3} \right], \quad a_{2n} = \frac{1}{32} (n^3 + 9n^2 + 22n + 32). \quad (A2)$$

Using Equations (48), one can obtain the expressions for the coefficients A through a :

$$\begin{aligned} A_{1n} &= 2(a_{1n+1} - a_{0n+1}), \quad A_{2n} = 2(A_{1n+1} - a_{1n+1}), \quad A_{3n} = 2(a_{2n+1} - a_{1n+1}), \\ A_{4n} &= 2(A_{3n+1} - A_{1n+1}), \quad A_{5n} = 3A_{4n} - 4A_{3n}, \quad A_{7n} = A_{3n} - A_{4n}/2, \quad A_{6n} = a_{2n} - 3A_{7n}. \end{aligned} \quad (A3)$$

The coefficients for $n = 0, 1, 2$ are presented in Table 1.

B. AUXILIARY FUNCTIONS ψ_{ij} AND Ψ_{ij}

The total cross-section and mean powers of energy of scattered photons are expressed through the functions of one variable

$$\psi_{ij}(\xi) = \frac{i+1}{\xi^{i+1}} \int_0^\xi x^i s_j(x) dx, \quad i > -1, \quad \psi_{-1j}(\xi) = \frac{1}{\xi} \int_0^\xi [1 - s_j(x)] \frac{dx}{x}, \quad \Psi_{ij}(\xi) = \frac{i+1}{\xi^{i+1}} \int_0^\xi x^i S_j(x) dx. \quad (B1)$$

Calculations of functions ψ_{ij} involve integrals of the following types:

$$\int dx x^m \ln(1+2x), \quad \int dx \frac{x^n}{(1+2x)^l}, \quad g(\xi) = \int_0^\xi \ln(1+2x) \frac{dx}{x}, \quad (B2)$$

where $m = -1, 0, 1, 2, 3$ and $n, l = 1, 2, 3, 4$. All integrals are elementary except $g(\xi)$, which is described in details in Appendix C.

The explicit expressions for the functions ψ_{ij} are the following:

$$\begin{aligned} \psi_{-10}(\xi) &= \frac{1}{8\xi} \left[\frac{4}{\xi^2} + \frac{2}{\xi} + \left(8 + \frac{3}{\xi} - \frac{3}{\xi^2} - \frac{2}{\xi^3} \right) l_\xi - 3R_\xi - \frac{11}{3} \right], \\ \psi_{-11}(\xi) &= \frac{1}{8\xi} \left[\frac{7}{3} + \frac{2}{\xi} - \frac{2}{\xi^2} + \left(8 + \frac{1}{\xi^3} \right) l_\xi - 4R_\xi - R_\xi^2 \right], \\ \psi_{-12}(\xi) &= \frac{1}{16\xi} \left[11 + 16l_\xi - 7R_\xi - 3R_\xi^2 - R_\xi^3 \right], \\ \psi_{00}(\xi) &= \frac{3}{8\xi} \left[g(\xi) - \frac{2}{\xi} + \left(\frac{1}{2} + \frac{2}{\xi} + \frac{1}{\xi^2} \right) l_\xi - \frac{1}{2}R_\xi - \frac{3}{2} \right], \\ \psi_{01}(\xi) &= \frac{3}{8\xi} \left[\frac{1}{\xi} - \frac{1}{2} + \left(\frac{4}{3} - \frac{1}{2\xi^2} \right) l_\xi - \frac{1}{3}R_\xi - \frac{1}{6}R_\xi^2 \right], \\ \psi_{02}(\xi) &= \frac{1}{32\xi} \left[\frac{7}{2} + 9l_\xi - R_\xi - \frac{3}{2}R_\xi^2 - R_\xi^3 \right], \\ \psi_{10}(\xi) &= \frac{3}{4\xi^2} \left[\left(\xi + \frac{9}{2} + \frac{2}{\xi} \right) l_\xi - 4 - \xi + \xi^2 R_\xi - 2g(\xi) \right], \\ \psi_{11}(\xi) &= \frac{1}{4\xi^2} \left[6 + 4\xi - 3 \left(\frac{3}{2} + \frac{1}{\xi} \right) l_\xi - \xi(1+\xi)R_\xi^2 \right], \\ \psi_{12}(\xi) &= \frac{1}{16\xi^2} \left[\frac{1}{2} + 9\xi - 4l_\xi - R_\xi + \frac{1}{2}R_\xi^3 \right], \\ \psi_{20}(\xi) &= \frac{9}{32\xi^3} \left[25\xi + (2\xi^2 - 8\xi - 5) l_\xi + \xi R_\xi - 8g(\xi) \right], \\ \psi_{21}(\xi) &= \frac{9}{8\xi^3} \left[g(\xi) + \frac{2}{3}\xi^2 - \frac{3}{2}\xi - \frac{1}{4}l_\xi - \frac{\xi^2}{6}R_\xi^2 \right], \\ \psi_{22}(\xi) &= \frac{3}{64\xi^3} \left[9\xi^2 - 8\xi + 5l_\xi - \xi(2+11\xi+10\xi^2)R_\xi^3 \right], \\ \psi_{30}(\xi) &= \frac{3}{12\xi^4} \left[\frac{\xi^3}{3} + \frac{31}{2}\xi^2 + \frac{31}{4}\xi + \left(2\xi^3 - 6\xi^2 - 12\xi - \frac{7}{2} \right) l_\xi - \frac{3}{4}\xi R_\xi \right], \\ \psi_{31}(\xi) &= \frac{1}{24\xi^4} \left[16\xi^3 - 27\xi^2 - 45\xi + 3(7+12\xi)l_\xi + 3\xi(1+3\xi)R_\xi^2 \right], \\ \psi_{32}(\xi) &= \frac{1}{16\xi^4} \left[6\xi^3 - 4\xi^2 + 5\xi - 3l_\xi + \xi(1+4\xi+2\xi^2)R_\xi^3 \right], \end{aligned} \quad (B3)$$

$$\begin{aligned}\psi_{40}(\xi) &= \frac{15}{128\xi^5} \left[\xi^4 + \frac{230}{9}\xi^3 + \frac{23}{6}\xi^2 - \frac{17}{6}\xi + \left(4\xi^4 - \frac{32}{3}\xi^3 - 16\xi^2 + \frac{11}{12} \right) l_\xi + \xi R_\xi \right], \\ \psi_{41}(\xi) &= \frac{5}{64\xi^5} \left[8\xi^4 - 12\xi^3 - 9\xi^2 + 8\xi + 3(4\xi^2 - 1) l_\xi - \xi(2 + 5\xi) R_\xi^2 \right], \\ \psi_{42}(\xi) &= \frac{5}{256\xi^5} \left[18\xi^4 - \frac{32}{3}\xi^3 + 10\xi^2 - 12\xi + \frac{11}{2} l_\xi + \xi(1 + 7\xi + 14\xi^2) R_\xi^3 \right],\end{aligned}$$

where $l_\xi = \ln(1 + 2\xi)$ and $R_\xi = 1/(1 + 2\xi)$.

The explicit expressions for Ψ_{ij} can be obtained using definitions (48) for S_j :

$$\begin{aligned}\Psi_{11} &= 2(\psi_{00} - \psi_{01})/\xi, & \Psi_{13} &= 2(\psi_{01} - \psi_{02})/\xi, & \Psi_{14} &= 2(2\psi_{-11} - \psi_{-10} - \psi_{-12})/\xi, \\ \Psi_{16} &= \psi_{12} - 3\Psi_{13} + 3\Psi_{14}/2, & \Psi_{21} &= 3(\psi_{10} - \psi_{11})/2\xi, & \Psi_{22} &= 3(\psi_{11} - \Psi_{11})/2\xi, \\ \Psi_{23} &= 3(\psi_{11} - \psi_{12})/2\xi, & \Psi_{24} &= 3(\Psi_{11} - \Psi_{13})/2\xi, & \Psi_{25} &= 3\Psi_{24} - 4\Psi_{23}, \\ \Psi_{26} &= \psi_{22} - 3\Psi_{23} + 3\Psi_{24}/2, & \Psi_{31} &= 4(\psi_{20} - \psi_{21})/3\xi, & \Psi_{32} &= 4(\psi_{21} - \Psi_{21})/3\xi, \\ \Psi_{33} &= 4(\psi_{21} - \psi_{22})/3\xi, & \Psi_{34} &= 4(\Psi_{21} - \Psi_{23})/3\xi, & \Psi_{35} &= 3\Psi_{34} - 4\Psi_{33}, \\ \Psi_{37} &= \Psi_{33} - \Psi_{34}/2, & \Psi_{36} &= \psi_{32} - 3\Psi_{37}, & \Psi_{41} &= 5(\psi_{30} - \psi_{31})/4\xi, \\ \Psi_{42} &= 5(\psi_{31} - \Psi_{31})/4\xi, & \Psi_{43} &= 5(\psi_{31} - \psi_{32})/4\xi, & \Psi_{44} &= 5(\Psi_{31} - \Psi_{33})/4\xi, \\ \Psi_{45} &= 3\Psi_{44} - 4\Psi_{43}, & \Psi_{47} &= \Psi_{43} - \Psi_{44}/2, & \Psi_{51} &= 6(\psi_{40} - \psi_{41})/5\xi, \\ \Psi_{54} &= 6(\Psi_{41} - \Psi_{43})/5\xi, & \Psi_{57} &= 6(\psi_{41} - \psi_{42})/5\xi - \Psi_{54}/2.\end{aligned}\tag{B4}$$

In these formulae the argument ξ is omitted. For complete evaluation of these functions we need to compute 18 different functions ψ_{ij} given above.

To prevent the loss of accuracy if ξ is very small, we can use the series expansions (see NP94) that directly follow from the definitions (B1) and Taylor expansions (A1):

$$\psi_{ij}(\xi) = \sum_{n=0}^{\infty} a_{jn} (-2\xi)^n \frac{i+1}{i+n+1}, \quad i > -1, \quad \psi_{-1j}(\xi) = \sum_{n=0}^{\infty} a_{jn+1} (-2\xi)^n \frac{2}{n+1},\tag{B5}$$

$$\Psi_{ij}(\xi) = \sum_{n=0}^{\infty} A_{jn} (-2\xi)^n \frac{i+1}{i+n+1},\tag{B6}$$

with a_{jn} and A_{jn} given by Equations (A2) and (A3), respectively.

C. AUXILIARY FUNCTION $g(\xi)$

Calculations of function ψ_{ij} from Appendix B involve integral

$$g(\xi) = \int_0^\xi \ln(1 + 2x) \frac{dx}{x}.\tag{C1}$$

We repeat here for completeness the method of calculations of this integral from NP94. It is possible to write a relation between the values of this function on $\xi < 1/2$ and $\xi > 1/2$. Let us define for that the auxiliary function for $\xi \leq 1$

$$g_*(\xi) = g(\xi/2) = \int_0^\xi \ln(1 + x) \frac{dx}{x}.\tag{C2}$$

It can be presented by series

$$g_*(\xi) = \begin{cases} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\xi^n}{n^2} & \text{if } \xi \leq \xi_* < 1, \\ \frac{\pi^2}{12} + \ln 2 \ln \xi + \sum_{k=0}^{\infty} \frac{(1-\xi)^{k+2}}{k+2} \sum_{m=1}^{k+1} \frac{1}{2^m m} & \text{if } \xi_* \leq \xi \leq 1. \end{cases}\tag{C3}$$

As ξ_* we can take $0.8 - 0.9$. Then

$$g(\xi) = \begin{cases} g_*(2\xi) & \text{if } 0 \leq \xi \leq 1/2, \\ \frac{\pi^2}{12} & \text{if } \xi = 1/2, \\ \frac{\pi^2}{6} + \frac{1}{2} \ln^2(2\xi) - g_*(1/2\xi) & \text{if } \xi \geq 1/2. \end{cases}\tag{C4}$$

D. ASYMPTOTIC EXPANSIONS OF FUNCTIONS χ_{jn} AND Δ_{jn} IN THOMSON LIMIT

Using Taylor expansion (B5) of functions ψ_{0n} for small arguments, it is easy to get an expansion of functions χ_{0n} in Thomson limit $x\gamma \ll 1$:

$$\chi_{0n}(x, \gamma) = \gamma^n \sum_{l=0}^{\infty} (-2x\gamma)^l a_{0l} \kappa_{n+l+1}, \quad (D1)$$

where

$$\kappa_l = \frac{(1+\beta)^{l+1} - (1-\beta)^{l+1}}{2\beta(l+1)} = \sum_{k=0}^{\text{int}(l/2)} \frac{l! \beta^{2k}}{(2k+1)!(l-2k)!} = \frac{(1+\beta)^l}{l+1} \sum_{k=0}^l \frac{1}{[\gamma(1+\beta)]^{2k}} \quad (D2)$$

and $\text{int}(x)$ is the integer part of x . A few first functions are

$$\begin{aligned} \kappa_1 &= 1, & \kappa_2 &= 1 + \frac{1}{3}\beta^2, & \kappa_3 &= 1 + \beta^2, & \kappa_4 &= 1 + 2\beta^2 + \frac{1}{5}\beta^4, & \kappa_5 &= 1 + \frac{10}{3}\beta^2 + \beta^4, \\ \kappa_6 &= 1 + 5\beta^2 + 3\beta^4 + \frac{1}{7}\beta^6, & \kappa_7 &= 1 + 7\beta^2 + 7\beta^4 + \beta^6. \end{aligned} \quad (D3)$$

The first three terms of the expansion (D1) are as follows:

$$\chi_{0n}(x, \gamma) \approx \gamma^n \left[\kappa_{1+n} - 2x\gamma\kappa_{2+n} + \frac{26}{5}(x\gamma)^2 \kappa_{3+n} \right]. \quad (D4)$$

Function Δ_{00} coincides with χ_{00} , and functions Δ_{01} and Δ_{02} can be obtained using definitions (27) and expansion (D1). For Δ_{01} , we get

$$\Delta_{01} = -\beta \sum_{l=0}^{\infty} (-2x\gamma)^l a_{0l} \zeta_{l+1} \approx -\frac{\beta}{3} \left[1 - 4x\gamma + \frac{78}{5}(x\gamma)^2 \left(1 + \frac{\beta^2}{5} \right) \right], \quad (D5)$$

where

$$\begin{aligned} \zeta_l &= \frac{\kappa_{l+1} - \kappa_l}{\beta^2} = \sum_{k=0}^{\text{int}[(l-1)/2]} \frac{2k+2}{(2k+3)!} \beta^{2k} \frac{l!}{(l-1-2k)!}, \\ \zeta_1 &= \frac{1}{3}, \quad \zeta_2 = \frac{2}{3}, \quad \zeta_3 = 1 + \frac{1}{5}\beta^2, \quad \zeta_4 = \frac{4}{3} + \frac{4}{5}\beta^2, \quad \zeta_5 = \frac{5}{3} + 2\beta^2 + \frac{1}{7}\beta^4, \quad \zeta_6 = 2 + 4\beta^2 + \frac{6}{7}\beta^4. \end{aligned} \quad (D6)$$

Respectively for Δ_{02} , we have

$$\Delta_{02} = \beta^2 \sum_{l=0}^{\infty} (-2x\gamma)^l a_{0l} \Lambda_{l+1} \approx -\frac{4}{15} \beta^2 (x\gamma) \left(1 - \frac{39}{5} x\gamma \right), \quad (D7)$$

where

$$\begin{aligned} \Lambda_l &= \frac{1}{2\beta^2} \left[3 \frac{\kappa_l - 2\kappa_{l+1} + \kappa_{l+2}}{\beta^2} - \kappa_l \right] = \sum_{k=1}^{\text{int}(l/2)} \frac{l!}{(l-2k)!(2k)!} \frac{2k}{(2k+1)(2k+3)} \beta^{2(k-1)}, \\ \Lambda_1 &= 0, \quad \Lambda_2 = \frac{2}{15}, \quad \Lambda_3 = \frac{2}{5}, \quad \Lambda_4 = \frac{4}{5} + \frac{4}{35}\beta^2. \end{aligned} \quad (D8)$$

Similarly, for functions χ_{1n} , using expansions (B6) we get:

$$\chi_{1n}(x, \gamma) = \gamma^{n+1} \sum_{l=0}^{\infty} (-2x\gamma)^l (\gamma A_{1l} + x A_{2l}) \kappa_{n+l+2} = \gamma^n \left[\gamma^2 \kappa_{2+n} + \sum_{l=1}^{\infty} (-2x\gamma)^l \left(\gamma^2 A_{1l} \kappa_{n+l+2} - \frac{A_{2l-1}}{2} \kappa_{n+l+1} \right) \right]. \quad (D9)$$

Functions Δ_{1n} can then be obtained using definitions (50):

$$\begin{aligned} \Delta_{10} &= \chi_{10} \approx 1 + \frac{4}{3} p^2 - \frac{x\gamma}{5} (42\gamma^2 - 27 - 2\beta^2), \\ \Delta_{11} &= -p \sum_{l=0}^{\infty} (-2x\gamma)^l (\gamma A_{1l} + x A_{2l}) \zeta_{l+2} \approx -\beta \left\{ \frac{2}{3} \gamma^2 - \frac{x\gamma}{5} [21\gamma^2(1 + \beta^2/5) - 4] \right\}, \\ \Delta_{12} &= p\beta \sum_{l=0}^{\infty} (-2x\gamma)^l (\gamma A_{1l} + x A_{2l}) \Lambda_{l+2} \approx \frac{2}{15} \beta^2 \left[\gamma^2 - \frac{3x\gamma}{5} (21\gamma^2 - 2) \right]. \end{aligned} \quad (D10)$$

For functions χ_{2n} , we can write the expansion

$$\begin{aligned}\chi_{2n}(x, \gamma) &= \gamma^n \sum_{l=0}^{\infty} (-2x\gamma)^l \left[(\gamma^4 A_{4l} - \gamma^2 A_{7l}) \kappa_{n+l+3} - \gamma^2 A_{5l} \kappa_{n+l+2} + A_{6l} \kappa_{n+l+1} \right] \\ &\approx \gamma^n \left[\frac{7}{5} \gamma^4 \kappa_{3+n} - \frac{\gamma^2}{10} (3\kappa_{3+n} + 2\kappa_{2+n}) + \frac{1}{10} \kappa_{1+n} \right],\end{aligned}\quad (\text{D11})$$

where we kept only the zeroth term in $x\gamma$ of the series. Expansions for Δ_{2n} can then be obtained using definitions (50):

$$\begin{aligned}\Delta_{20} &= \chi_{20} \approx 1 + \frac{2}{15} p^2 (21\gamma^2 + 4), \\ \Delta_{21} &= -\beta \sum_{l=0}^{\infty} (-2x\gamma)^l \left[(\gamma^4 A_{4l} - \gamma^2 A_{7l}) \zeta_{l+3} - \gamma^2 A_{5l} \zeta_{l+2} + A_{6l} \zeta_{l+1} \right] \approx -\beta \left[1 + \frac{2}{75} p^2 (63\gamma^2 + 34) \right], \\ \Delta_{22} &= \beta^2 \sum_{l=0}^{\infty} (-2x\gamma)^l \left[(\gamma^4 A_{4l} - \gamma^2 A_{7l}) \Lambda_{l+3} - \gamma^2 A_{5l} \Lambda_{l+2} + A_{6l} \Lambda_{l+1} \right] \approx p^2 \frac{1}{75} (42\gamma^2 - 11).\end{aligned}\quad (\text{D12})$$

Let us now discuss the properties of functions χ_{1n}^* . The series expansion can be easily obtained from the definition (89) and series (B6):

$$\chi_{1n}^*(x, \gamma) = \gamma^{n+2} \sum_{l=0}^{\infty} (-2x\gamma)^l A_{1l} \kappa_{n+l+3}. \quad (\text{D13})$$

For Δ_{1k}^* we get:

$$\begin{aligned}\Delta_{10}^* &= \chi_{10}^* \approx \gamma^2 (1 + \beta^2), \\ \Delta_{11}^* &= -\gamma p \sum_{l=0}^{\infty} (-2x\gamma)^l A_{1l} \zeta_{l+3} \approx -\gamma p (1 + \beta^2/5), \\ \Delta_{12}^* &= p^2 \sum_{l=0}^{\infty} (-2x\gamma)^l A_{1l} \Lambda_{l+3} \approx \frac{2}{5} p^2.\end{aligned}\quad (\text{D14})$$

The series expansion for functions χ_{1n}^\perp are:

$$\chi_{1n}^\perp(x, \gamma) = \frac{\gamma^{n+1}}{p} \sum_{l=0}^{\infty} (-2x\gamma)^l A_{1l} (\kappa_{n+l+2} - 2\gamma^2 \kappa_{n+l+3} + \gamma^2 \kappa_{n+l+4}) = \frac{2}{3} \gamma^{n+1} p \sum_{l=0}^{\infty} (-2x\gamma)^l A_{1l} (\beta^2 \Lambda_{n+l+2} - \kappa_{n+l+2}). \quad (\text{D15})$$

For Δ_{1k}^\perp we get:

$$\begin{aligned}\Delta_{11}^\perp &= \frac{1}{2} \chi_{10}^\perp \approx -\frac{1}{3} \gamma p (1 + \beta^2/5), \\ \Delta_{12}^\perp &= \frac{1}{2p} (\gamma \chi_{10}^\perp - \chi_{11}^\perp) = \frac{1}{3} p^2 \sum_{l=0}^{\infty} (-2x\gamma)^l A_{1l} [\Lambda_{l+2} - \Lambda_{l+3} + \zeta_{l+2}] \approx \frac{2}{15} p^2.\end{aligned}\quad (\text{D16})$$

E. ELIMINATING CANCELLATIONS IN REDISTRIBUTION FUNCTIONS

If formulae (138), (139) and (140) are used as they stand, numerical cancellations appear at certain regions of parameter space. For example if x and x_1 are small, the quantities a_- and a_+ , $1/a_-$ and $1/a_+$, are close to each other. Also a combination containing a sum of d_-/a_-^3 and d_+/a_+^3 minus double the difference $1/a_-$ and $1/a_+$ has a cancellation. Therefore it is useful to rewrite the expressions in a form not containing those cancellations. The cancellations appearing in (138) were dealt with in Nagirner & Poutanen (1993). Defining

$$u = a_+ - a_- = \frac{(x + x_1)(2\gamma + x_1 - x)}{a_- + a_+}, \quad v = a_- a_+, \quad (\text{E1})$$

they got

$$R_0 = \frac{2}{Q} + \frac{u}{v} \left(1 - \frac{2}{q} \right) + u \frac{(u^2 - Q^2)(u^2 + 5v)}{2q^2 v^3} + u \frac{Q^2}{q^2 v^2}. \quad (\text{E2})$$

Using definitions (E1), we get from (139) and (140)

$$R_\Sigma = (a_- + a_+) \left[\frac{2u}{Q^3} + \frac{1}{v} \left(1 - \frac{2}{q} \right) + \frac{(u^2 - Q^2)(u^2 + 3v)}{2q^2 v^3} + \frac{Q^2}{q^2 v^2} \right], \quad (\text{E3})$$

$$R_{\Pi} = \frac{1}{2Q^5} [u^2(u^2 + 4v) + 2b^2] + \frac{1}{2Q} \left(1 - \frac{4}{q}\right) + \frac{u}{q^2 v}. \quad (\text{E4})$$

Another loss of accuracy occurs in $u^2 - Q^2$ term, when γ is close to $\gamma_*(x, x_1, \mu)$. We can use the following formulae (Nagirner & Poutanen 1993):

$$u^2 - Q^2 = 2rqCD_u, \quad D_u = (\gamma + x_1 - x + \gamma_*)(\gamma - \gamma_*), \quad C = 2/[\gamma(\gamma + x_1 - x) + r + xx_1\mu + v]. \quad (\text{E5})$$

F. BOUNDARIES

The redistribution functions R_0, R_{Σ}, R_{Π} and R_1, R_2 are defined within the interval of photon and electron energies and scattering angles satisfying the relation $|\cos\theta| \leq 1$, where $\cos\theta$ is given by Equation (107). These limits were discussed in NP94, but we repeat them here for completeness. For fixed photon energies and scattering angle, we already got the limits on the electron energies given by Equation (110), $\gamma \geq \gamma_*(x, x_1, \mu)$. If we are interested in the interval of scattered photon energies for the fixed x_1, γ and μ , we have then $x^- \leq x \leq x^+$, where

$$x^{\pm}(x_1, \gamma, \mu) = x_1 \frac{\mu + \gamma(\gamma + x_1)(1 - \mu) \pm p(1 - \mu)a_+}{1 + 2\gamma x_1(1 - \mu) + x_1^2(1 - \mu)^2}. \quad (\text{F1})$$

If the energy of the scattered photon x is fixed, the initial photon x_1 lies in the interval

$$\begin{aligned} x_1^- \leq x_1 \leq x_1^+ & \quad \text{if } 0 \leq x(1 - \mu) \leq \gamma - p, \\ x_1 > x_1^- & \quad \text{if } \gamma - p \leq x(1 - \mu) \leq \gamma + p, \end{aligned} \quad (\text{F2})$$

where

$$x_1^{\pm}(x, \gamma, \mu) = x \frac{\mu + \gamma(\gamma - x)(1 - \mu) \pm p(1 - \mu)a_-}{1 - 2\gamma x(1 - \mu) + x^2(1 - \mu)^2}. \quad (\text{F3})$$

In Equations (F1) and (F3), the quantities a_{\pm} are defined by Equations (119). If $|x - x_1| \leq 2xx_1$, the quantity $\gamma_*(x, x_1, \mu)$ as a function of μ has a minimum

$$\gamma_{\min} = 1 + (x - x_1 + |x - x_1|)/2 \quad (\text{F4})$$

at $\mu = \mu_{\min} = 1 - |x - x_1|/xx_1$, while in the opposite case, $|x - x_1| \geq 2xx_1$, the function is monotonic with the minimum reached at the boundary $\mu = -1$ (see Fig. 12). Correspondingly, the limits of variations of μ depend on the photon energies x, x_1 and the electron energy γ and are given by

$$\mu_{\min} \leq \mu \leq \mu_{\max}, \quad (\text{F5})$$

where

$$\begin{aligned} \mu_{\min}(x, x_1, \gamma) &= \begin{cases} -1 & \text{if } |x - x_1| \geq 2xx_1, \\ -1 & \text{if } |x - x_1| \leq 2xx_1 \text{ and } \gamma \geq \gamma_*(x, x_1, -1), \\ \mu_- & \text{if } |x - x_1| \leq 2xx_1 \text{ and } \gamma \leq \gamma_*(x, x_1, -1), \end{cases} \\ \mu_-(x, x_1, \gamma) &= 1 - \frac{q_+}{xx_1}, \\ \mu_+(x, x_1, \gamma) &= 1 - \frac{q_-}{xx_1} = 1 - \frac{(x - x_1)^2}{xx_1 q_+}, \end{aligned} \quad (\text{F6})$$

and

$$\gamma_*(x, x_1, -1) = [x - x_1 + (x + x_1)\sqrt{1 + 1/xx_1}]/2, \quad (\text{F7})$$

$$q_{\pm} = p^2 + \gamma(x_1 - x) \pm p\sqrt{(\gamma + x_1 - x)^2 - 1}. \quad (\text{F8})$$

For the angle-averaged redistribution function, the lower limit on the electron energy is:

$$\gamma_{\star}(x, x_1) = \begin{cases} \gamma_*(x, x_1, -1) & \text{if } |x - x_1| \geq 2xx_1, \\ \gamma_{\min} & \text{if } |x - x_1| \leq 2xx_1. \end{cases} \quad (\text{F9})$$

The limits of variation of the scattered photon energy x as a function of incident photon energy x_1 and γ can be found by inverting Equation (F9). We obtain

$$x^-(x_1, \gamma) \leq x \leq x_{\min}(x_1, \gamma), \quad (\text{F10})$$

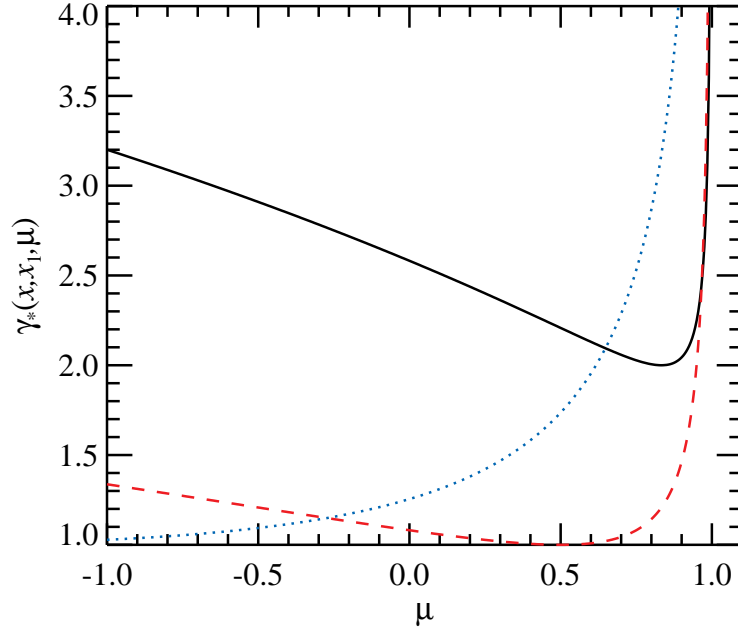


FIG. 12.— The dependence of the function $\gamma_*(x, x_1, \mu)$ on μ . The energy of the incident photon is $x_1 = 2$. The solid curves are for $x = 3$ and the dashed curves are for $x = 1$ (both cases correspond to $|x - x_1| < 2xx_1$), and the dotted curve is for $x = 0.3$ (where $|x - x_1| > 2xx_1$).

where

$$x_m(x_1, \gamma) = \begin{cases} \gamma + x_1 - 1 & \text{if } 1 \leq \gamma \leq 1 + \frac{2x_1^2}{1 - 2x_1} \text{ and } x_1 < 1/2, \\ x^+(x_1, \gamma) & \text{if } 1 + \frac{2x_1^2}{1 - 2x_1} \leq \gamma \text{ and } x_1 < 1/2, \\ \gamma + x_1 - 1 & \text{if } x_1 \geq 1/2, \end{cases}$$

$$x^\pm(x_1, \gamma) = x_1 (\gamma \pm p) / (\gamma \mp p + 2x_1). \quad (\text{F11})$$

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